SUGGESTED SOLUTION TO PROBLEM 11

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This is a suggested solution for Problem 11. If you find something that looks like a typo or error (or if you have questions, or want additional feedback on an attempted solution), feel free to email me.

Let $\mathbb{R}^n$ be equipped with the standard topology, let $\|\cdot\|$ be the Euclidean norm, and let $\sim$ be the equivalence relation on $\mathbb{R}^n$ defined by setting $x \sim y$ if and only if $\|x\| = \|y\|$.

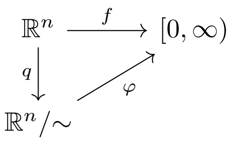
Claim 1: $\mathbb{R}^n/{\sim}$ (with the quotient topology) is homeomorphic to $[0, \infty)$.

Proof. Consider the map

$$f\colon \mathbb{R}^n \to [0,\infty)\,, \quad x \mapsto \|x\|\,.$$

We know that this map is continuous from analysis (but it's also pretty easy to check that $f^{-1}((a,b)) \subseteq \mathbb{R}^n$ and $f^{-1}([0,b)) \subseteq \mathbb{R}^n$ are open for all $0 \leqslant a < b$). Since f is constant on equivalence classes, the ***universal property of quotient spaces*** (Proposition 7.12 in the script, see also the end of these notes) gives that it induces a continuous map $\varphi\colon \mathbb{R}^n/{\sim} \to [0,\infty)$ given by $\varphi([x]) = f(x) = \|x\|$.

Sidenote for those who are interested: In fancy category-theoretic language, one can say that φ makes the following diagram commute:



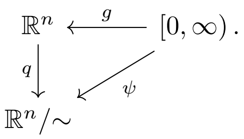
where $q\colon \mathbb{R}^n \to \mathbb{R}^n/{\sim}$ is the quotient map defined by $q(x) = [x]$.

We now claim that φ is a homeomorphism. We show this by constructing a continuous inverse. Consider the map

$$g\colon [0,\infty) \to \mathbb{R}^n\,, \quad g(t) = (t, 0, \ldots, 0)\,.$$

This is a continuous map, which we again can argue for by either referring to analysis, or by checking that $g^{-1}(B_r(x)) \subseteq \mathbb{R}^n$ is open for all $r > 0$ and $x \in \mathbb{R}^n$. Form the composition $\psi = q \circ g$, given by $\psi(t) = [(t, 0, \ldots, 0)]$. Note that ψ is continuous since it's the composition of two continuous maps.

With fancy language, one can say that it makes the following diagram commute:



It now holds that

$$\varphi(\psi(t)) = \varphi([(t,0,\ldots,0)]) = \|(t,0,\ldots,0)\| = t \quad \text{for } t \in [0,\infty)\,,$$
$$\psi(\varphi([x])) = \psi(\|x\|) = [(\|x\|,0,\ldots,0)] = [x] \quad \text{for } x \in \mathbb{R}^n\,,$$

where the last equality holds since $\|(\|x\|, 0, \ldots, 0)\| = \|x\|$. This shows that ψ is the inverse of φ. □

Claim 2: $[0,\infty)$ is homeomorphic to $[0, 1)$.

Proof. Simply note that the map

$$\alpha\colon [0,\infty) \to [0,1)\,, \quad t \mapsto \frac{t}{t+1}\,,$$

is continuous (we know from analysis that rational functions are continuous) and invertible, with continuous inverse

$$\beta\colon [0,1) \to [0,\infty)\,, \quad s \mapsto \frac{s}{1-s}\,.$$ □

Main claim: $\mathbb{R}^n/{\sim}$ is homeomorphic to $[0,1)$.

Proof. This follows from Claim 1 and Claim 2, combined with the fact that being homeomorphic is a transitive relation for topological spaces (see Remark 6.5 in the script). More specifically, the composition $\alpha \circ \varphi\colon \mathbb{R}^n/{\sim} \to [0,1)$ will be a homeomorphism, with continuous inverse $\psi \circ \beta\colon [0,1) \to \mathbb{R}^n/{\sim}$:

$$\mathbb{R}^n/{\sim} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} [0,\infty) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} [0,1)\,.$$

□

The universal property of quotient spaces (special case of Proposition 7.12)**.** Let X and Z be topological spaces, let $\sim$ be an equivalence relation on X, let $X/{\sim}$ be equipped with the quotient topology, and let $q\colon X \to X/{\sim}$ be the quotient map defined by $q(x) = [x]$. Let $f\colon X \to Z$ be a continuous map that is constant on equivalence classes, in the sense that $z_1 \sim z_2$ implies $f(z_1) = f(z_2)$. Then there is a unique continuous map $\varphi\colon X/{\sim} \to Z$ such that $\varphi([x]) = f(x)$ for all $x \in X$, or put differently, that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Z\,. \\ {\scriptstyle q}\downarrow & \nearrow_{\varphi} & \\ X/{\sim} & & \end{array}$$

LaTeX advice. To get the spacing in $X/{\sim}$ right, you should write `X/{\sim}`, rather than `X/\sim`.