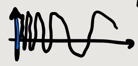
Lecture 11: Intro to compactness

Last time: connectedness $\overset{\Leftarrow}{\not\Rightarrow}$ path connectedness

counterexample



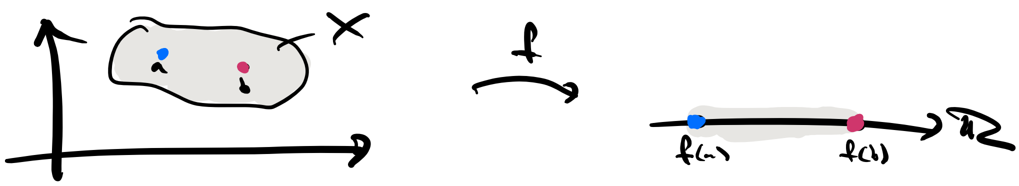
Today: Compactness (§9)

One motivation: Generalize the extreme value theorem:

If $X \subseteq \mathbb{R}^n$ (with Euclidean metric) is closed and bounded,
then any continuous map $f: X \to \mathbb{R}$
has a min and a max value:

$$\exists a \in X \text{ such that } \forall x \in X: f(x) \geq f(a)$$
$$\exists b \in X \text{ such that } \forall x \in X: f(x) \leq f(b).$$

Picture:



Note: • "Bounded" relies on X having a metric.

• In general metric spaces, closed and bounded is not sufficient for X to satisfy the EVT.

Ex: $X = \mathbb{Z}_{>0}$ in $\mathbb{R}$ with discrete metric

$f: X \to \mathbb{R}$
$n \mapsto \frac{1}{n}$

(see the exercises!)

Q: What should we replace "closed and bounded" by for general spaces X not embedded in $\mathbb{R}^n$?

Def. A top sp $(X, \mathcal{T})$ is compact if

$\forall \mathcal{A} \subseteq \mathcal{T}$ with $\bigcup_{U \in \mathcal{A}} U = X$ $\exists n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{A}$ such that $\bigcup_{i=1}^{n} U_i = X$

every open covering ... ... has a finite subcovering

Def. A subset $Y \subseteq X$ is called compact if it is compact with subspace topology in X.

Equivalently: $\forall \mathcal{A} \subseteq \mathcal{T}$ with $Y \subseteq \bigcup_{U \in \mathcal{A}} U$ $\exists n \in \mathbb{Z}_{>0}$, $U_1, \ldots, U_n \in \mathcal{A}$ s.t. $Y \subseteq \bigcup_{i=1}^{n} U_i$

Think-pair-share: Compact or not?

(a) $X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{standard}}$

Not compact: $\mathcal{A} = \{(-n, n) : n \in \mathbb{Z}_{>0}\}$ has no finite subcovering

(b) $X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{discrete}} \leftarrow \mathcal{P}(X)$

Not compact: $\mathcal{T}_{\text{standard}} \subseteq \mathcal{T}_{\text{discrete}}$

A finer topology makes it harder to be compact!

(c) $X = \mathbb{R} \leftarrow$ or any set! $\mathcal{T} = \mathcal{T}_{\text{trivial}} \leftarrow \{\emptyset, X\}$

Compact: $\mathcal{A} = \{\emptyset, X\}$ or $\mathcal{A} = \{X\}$ are the only open coverings, and they are already finite!

(d) $X = \mathbb{R} \leftarrow$ or any set! $\mathcal{T} = \mathcal{T}_{\text{cofinite}} \leftarrow \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ finite}\}$

Compact: Let $\mathcal{A} \subseteq \mathcal{T}_{\text{cofinite}}$ with $\bigcup_{U \in \mathcal{A}} U = X$.

There must exist nonempty $U_0 \in \mathcal{A}$ with $U_0 = X \setminus \{x_1, \ldots, x_m\}$ for some $m \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_m \in X$.

For each $i \in \{1, \ldots, m\}$, there is some $U_i \in \mathcal{A}$ s.t. $x_i \in U_i$.

Then $U_0 \cup U_1 \cup \cdots \cup U_m = X$.

Generalization of (c): Any space $(X, \mathcal{T})$ with finite $\mathcal{T}$ is compact.

In particular, this is true if X is finite ($\#\mathcal{T} \leq 2^{\#X}$).

Heine-Borel thm: $X \subseteq \mathbb{R}^n$ with subspace topology. Then

$$[X \text{ is compact}] \Longleftrightarrow [X \text{ is closed and bounded}].$$

Pf. Next lecture!

Ex: $[0,1] \subseteq \mathbb{R}$ is compact

$(0,1) \subseteq \mathbb{R}$ is not compact

$A = \{(\frac{1}{n}, 1) : n \in \{2,3,4,\dots\}\}$ has no finite subcovering

Prop. Let $f: X \to Y$ be a cont map. Then

$$[X \text{ is compact}] \Rightarrow [f(X) \text{ is compact}]$$

Pf. Let $f(X) \subseteq \bigcup_{V \in A} V$ for some $A \subseteq \mathcal{T}_Y$

Then $X = f^{-1}\left(\bigcup_{V \in A} V\right) = \bigcup_{V \in A} f^{-1}(V)$

$f^{-1}(V) \in \mathcal{T}_X$ bc f is cont

So $\{f^{-1}(V) : V \in A\}$ is open covering of X.

By compactness, $\exists n \in \mathbb{Z}_{>0}, V_1, \ldots, V_n \in \mathcal{A}$ s.t. $X = \bigcup_{i=1}^{n} f^{-1}(V_i)$

But then

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_i)\right) = \bigcup_{i=1}^{n} f(f^{-1}(V_i)) \subseteq \bigcup_{i=1}^{n} V_i$$

So we have found a finite subcovering of $\mathcal{A}$. $\square$

Cor. Let X be a top. sp. and $\sim$ an equivalence relation on X. Then

$$[X \text{ compact}] \Rightarrow [X/\sim \text{ compact}]$$

Pf. $X/\sim$ is the image of $\pi : X \to X/\sim$, $x \mapsto [x]$. $\square$

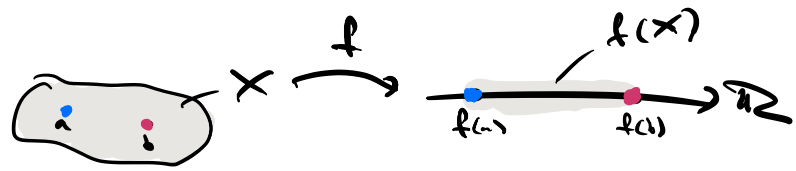
Extreme value thm.

Let X be compact, and $f : X \to \mathbb{R}$ (with standard topology) a cont. map.

Then (i) $\exists a \in X$ s.t. $\forall x \in X : f(x) \geq f(a)$

(ii) $\exists b \in X$ s.t. $\forall x \in X : f(x) \leq f(b)$.

Picture :



Pf. Note: $f(X) \subseteq \mathbb{R}$ is compact by previous prop.

(i) Suppose for a contradiction that $f(X) \subseteq \mathbb{R}$ has no smallest element. Then

$$\mathcal{A} = \{(z, \infty) : z \in f(X)\}$$ is an open covering of $f(X)$

By assumption: $\forall y \in f(X)\ \exists z \in f(X)$ with $y \in (z, \infty)$

By compactness, $\exists n \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_n \in f(X)$ such that
with $z_1 < z_2 < \cdots < z_n$

$$f(X) \subseteq \bigcup_{i=1}^{n} (z_i, \infty) = (z_1, \infty).$$

This is impossible, since $z_1 \in f(X)$ but $z_1 \notin (z_1, \infty)$. $\lightning$

<u>Conclusion</u>: $f(X)$ has a smallest element, say $w \in f(X)$, with $w = f(a)$ for some $a \in X$. It satisfies

$$\forall x \in X: \quad f(x) \geq w = f(a).$$

(ii) Analogous! $\square$

<u>Remark.</u> We can replace the codomain $\mathbb{R}$ by any totally ordered set Y with the order topology.

<u>Think-pair-share</u>:

lines through origin in $\mathbb{R}^3$

(a) Is $\mathbb{RP}^2 = \dfrac{\mathbb{R}^3 \setminus \{0\}}{x \sim \lambda x \ \ \forall x \in \mathbb{R}^3\setminus\{0\},\ \forall \lambda \in \mathbb{R}\setminus\{0\}}$ compact?

Yes! It's the image of $S^2 \subseteq \mathbb{R}^3$ under $\pi: \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$, $(x,y,z) \mapsto [x:y:z]$

closed and bounded so compact by Heine-Borel

(b) Does $f: \mathbb{RP}^2 \longrightarrow \mathbb{R}$, $[x:y:z] \longmapsto \dfrac{xyz}{x^3+y^3+z^3}$ have a min and max value?

Yes, follows by the extreme value thm, since f is continuous by the universal property of quotients.

(In fact, it's not hard to see that $\min_{x \in \mathbb{RP}^3} f(x) = -\frac{1}{3}$ and $\max_{x \in \mathbb{RP}^3} f(x) = \frac{1}{3}$.)

Next up: Connection to closedness and Hausdorffness

Thm. Let $(X,\mathcal{T})$ be a compact space, and $Y \subseteq X$. Then

$$[Y \text{ closed}] \Longrightarrow [Y \text{ compact}]$$

↖ in X

Pf. Suppose $Y \subseteq \bigcup_{U \in \mathcal{A}} U$ for some $\mathcal{A} \subseteq \mathcal{T}$.

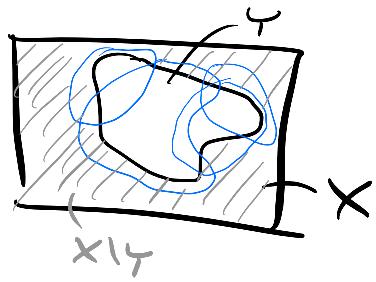
Trick: $X = (X \setminus Y) \cup \left(\bigcup_{U \in \mathcal{A}} U\right)$ is an open covering of X

↖ open b/c Y is closed

Compactness now gives that $\exists n \in \mathbb{Z}_{\geq 0}, U_1, \ldots, U_n \in \mathcal{A}$ s.t.

$$X = (X \setminus Y) \cup \left(\bigcup_{i=1}^{n} U_i\right)$$

$$\Longrightarrow Y \subseteq \bigcup_{i=1}^{n} U_i$$



i.e. we've found a finite subcovering! $\square$

⚠ The converse is not true in general!

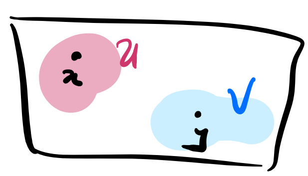
Ex: $X = \{a,b\}$, $\mathcal{T} = \mathcal{T}_{trivial}$, $Y = \{a\}$

X is compact, and Y is compact, but Y is not closed in X.

We will soon see that the converse is true if X is Hausdorff

Recall: $(X,\mathcal{T})$ is called Hausdorff if

$$\forall x,y \in X,\ x \neq y\ \ \exists U,V \in \mathcal{T} \text{ s.t. } U \cap V = \emptyset,\ x \in U,\ y \in V$$



<u>Generalization</u>: Let $(X,\mathcal{T})$ be Hausdorff. Then

$\forall x \in X$ $\forall$ compact $Y \subseteq X$ s.t. $x \notin Y$
$\exists\, \mathcal{U}, V \in \mathcal{T}$ s.t. $\mathcal{U} \cap V = \emptyset$, $x \in \mathcal{U}$ and $Y \subseteq V$.

<u>Pf.</u> For any $y \in Y$, we can choose

$\mathcal{U}_y$ neighborhood of x
V_y neighborhood of y s.t. $\mathcal{U}_y \cap V_y = \emptyset$

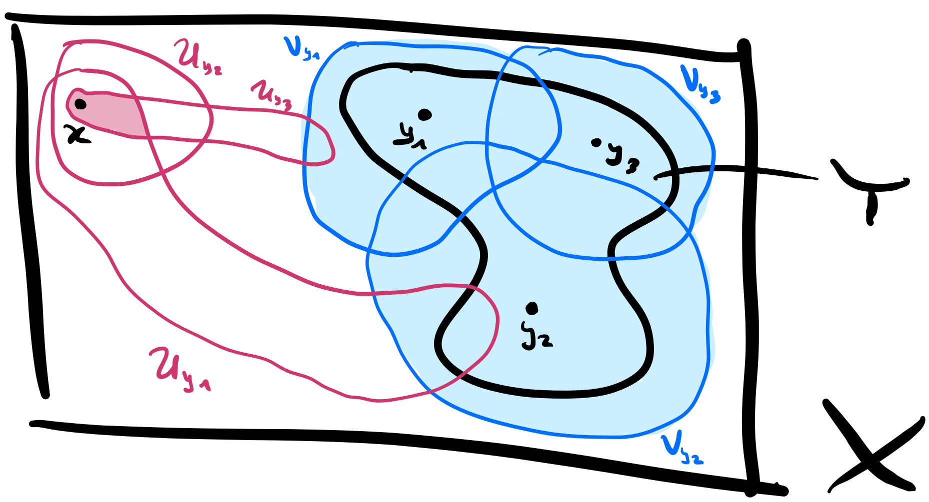
Then $\{V_y : y \in Y\}$ covers Y.

By compactness, $\exists\, n \in \mathbb{Z}_{>0}$, $y_1, \dots, y_n \in Y$ s.t. $Y \subseteq \bigcup_{i=1}^{n} V_{y_i}$

Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_{y_i}$. Then:

- V is open ← union of open sets
- $\mathcal{U}$ is open ← finite intersection of open sets
- $x \in \mathcal{U}$ ← $x \in \mathcal{U}_{y_i}$ $\forall i \in \{1, \dots, n\}$
- $\mathcal{U} \cap V = \mathcal{U} \cap \left(\bigcup_{i=1}^{n} V_{y_i}\right) = \bigcup_{i=1}^{n} \underbrace{(\mathcal{U} \cap V_{y_i})}_{\text{empty b/c } \mathcal{U} \subseteq \mathcal{U}_{y_i} \text{ and } \mathcal{U}_{y_i} \cap V_{y_i} = \emptyset} = \bigcup_{i=1}^{n} \emptyset = \emptyset.$ $\square$

<u>Picture</u>:



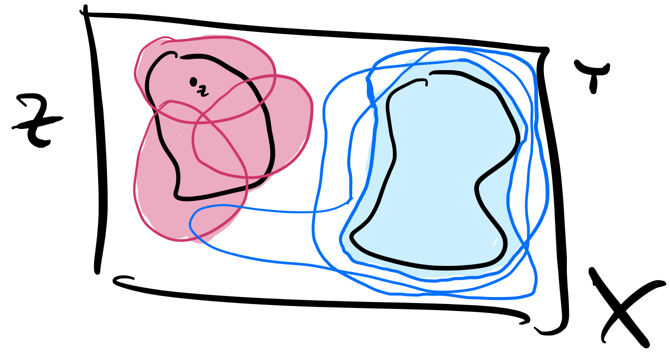
Suggestion: Draw the picture step by step while working through the proof!

Further generalization: Let $(X, \mathcal{T})$ be Hausdorff, and let $Z, Y \subseteq X$ be compact subsets with $Z \cap W = \emptyset$. Then

$$\exists\, \mathcal{U}, V \in \mathcal{T} \text{ s.t. } \mathcal{U} \cap V = \emptyset,\ Z \subseteq \mathcal{U},\ Y \subseteq V.$$

Pf. Exercise!

Picture:



Thm 9.10: Let X be Hausdorff, and $Y \subseteq X$. Then

$$[Y \text{ compact}] \implies [Y \text{ closed}].$$

Pf. Strategy: Show that $X \setminus Y$ is open.

For each $x \in X \setminus Y$, the generalized Hausdorff property gives that there are $\mathcal{U}_x, V_x \in \mathcal{T}$ s.t. $\mathcal{U}_x \cap V_x = \emptyset$, $x \in \mathcal{U}_x$ and $Y \subseteq V_x$.

Then in particular, $\mathcal{U}_x \subseteq X \setminus Y$, and we conclude $X \setminus Y = \bigcup_{x \in X \setminus Y} \mathcal{U}_x$ is open.

Prop 9.14: Let $f: X \to Y$ be a bijective cont. map, where X is compact and Y is Hausdorff. Then f is a homeomorphism.

Pf. Since, by assumption, f is bijective and cont., it suffices to show it's closed.

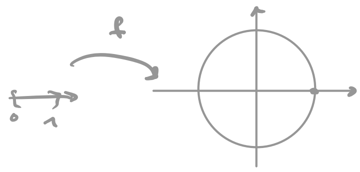
Let $A \subseteq X$ be closed. Then by previous results:

A is compact, $\quad f(A) \subseteq Y$ is compact, $\quad$ and $\quad f(A)$ is closed. $\square$

closed in compact space $\quad$ image of compact space under cont. map $\quad$ compact in Hausdorff space

Think-pair-share: Which are homeomorphisms?

$$f: [0,1] \longrightarrow S^1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}$$
$$t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$$

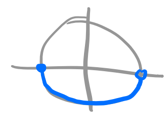


Not injective!

$f(0) = f(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$g: [0,1) \to S^1$$
$$t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$$

$g\left([\tfrac{1}{2}, 1) \right) \subseteq S^1$ is not closed



$$h: \frac{[0,1]}{0 \sim 1} \to S^1$$
$$[t] \longmapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$$

Homeomorphism!

- Bijective
- continuous by the universal property of quotients
- $[0,1]/\sim$ is compact b/c quotient of compact space
- S^1 is Hausdorff b/c subspace of $\mathbb{R}^2$.

Next time:

- Heine-Borel $\quad$ [compact in $\mathbb{R}^n$] $\Leftrightarrow$ [closed and bounded]
- Tychonoff $\quad$ [X_i compact $\forall i \in I$] $\Rightarrow$ [$\prod_{i \in I} X_i$ compact]
- Sequential compactness
- One-point Compactification $\quad X \rightsquigarrow X^+ := X \cup \{\infty\}$