Lecture 10: Wrapping up connectedness

Last time: connectedness
Today: $\Downarrow\!\!\!\!/ \quad \Uparrow$
path-connectedness

§ Revision of connectedness.

Def. A top sp $(X, \mathcal{T})$ is disconnected if
$\exists\, \mathcal{U}, V \in \mathcal{T}$ such that

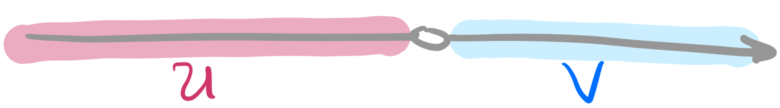
$$\mathcal{U} \neq \emptyset,\ V \neq \emptyset, \qquad \mathcal{U} \cap V = \emptyset, \qquad \mathcal{U} \cup V = X.$$

It's called connected if not disconnected.

Think-pair-share: Connected or disconnected?

$X = \mathbb{R} \setminus \{0\}$, $\mathcal{T}$ = subspace topology in $\mathbb{R}$

Disconnected: $\mathcal{U} = \mathbb{R}_{<0}$, $V = \mathbb{R}_{>0}$.



$X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{standard}}$

Connected: $X = \bigcup_{n=1}^{\infty} [-n, n]$ (see last week!)

$X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{cofinite}} = \{ \mathcal{U} \subseteq X : \mathcal{U} = \phi \text{ or } \#(X \setminus \mathcal{U}) < \infty \}$

Also connected: $\mathcal{T}_{\text{cofinite}} \subseteq \mathcal{T}_{\text{standard}}$

Making the topology coarser just makes it harder to find a disconnection!

$X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{trivial}} = \{ \phi, X \}$

Connected: Same reason!

$X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_{\text{discrete}} = \mathcal{P}(X)$

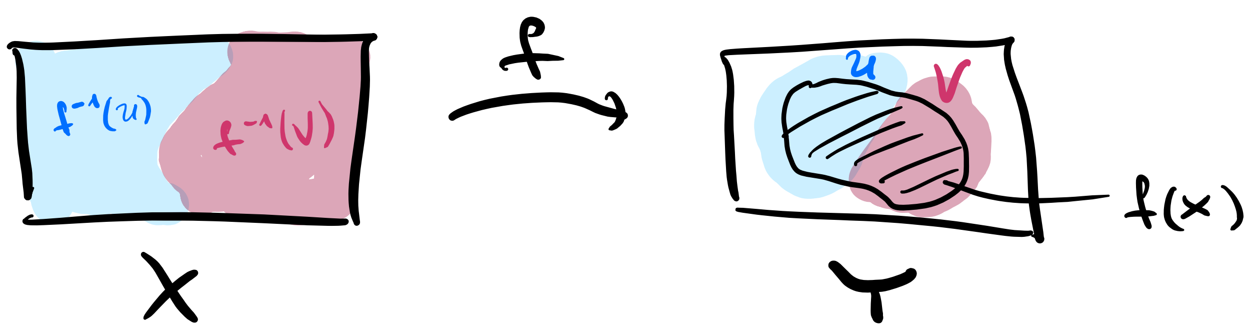
Disconnected: $\mathcal{U} = \mathbb{R}_{<0}$, $V = \mathbb{R}_{\geq 0}$.

Prop. Let $f : X \to Y$ be a continuous map. Then

$$[X \text{ connected}] \Rightarrow [f(X) \text{ connected}].$$

with subspace topology in Y

Picture:



Prop (Lemma 8.15). Let $(X, \mathcal{T}_X)$ be a top sp, and $A \subseteq X$. Then

$$[A \text{ connected}] \implies [\bar{A} \text{ connected}].$$

with subspace topology in X

Pf. Contraposition! Suppose $\bar{A}$ is disconnected. Then there exists $\mathcal{U}, V \in \mathcal{T}_X$ such that

$$\mathcal{U} \cap \bar{A} \neq \emptyset \qquad V \cap \bar{A} \neq \emptyset \qquad (\mathcal{U} \cap \bar{A}) \cap (V \cap \bar{A}) = \emptyset \qquad \bar{A} = (\mathcal{U} \cap \bar{A}) \cup (V \cap \bar{A})$$

property of closure

intersect both sides with A

intersect both sides with A

$$\mathcal{U} \cap A \neq \emptyset \qquad V \cap A \neq \emptyset \qquad (\mathcal{U} \cap A) \cap (V \cap A) = \emptyset \qquad A = (\mathcal{U} \cap A) \cup (V \cap A)$$

Conclude: A is disconnected! $\square$

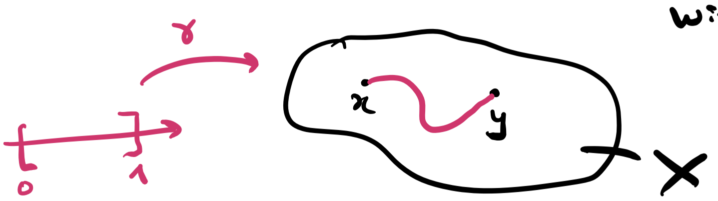
Ex: $X = \mathbb{R}$, $\mathcal{T}_X = \mathcal{T}_{\text{standard}}$, $A = (0,1)$, $\bar{A} = [0,1]$ both connected (see last week!)

§ Pathconnectedness

Def. A topological space $(X, \mathcal{T})$ is path connected if

$\forall x, y \in X \ \exists$ continuous map $\gamma : [0,1] \to X$

with $\gamma(0) = x$ and $\gamma(1) = y$.



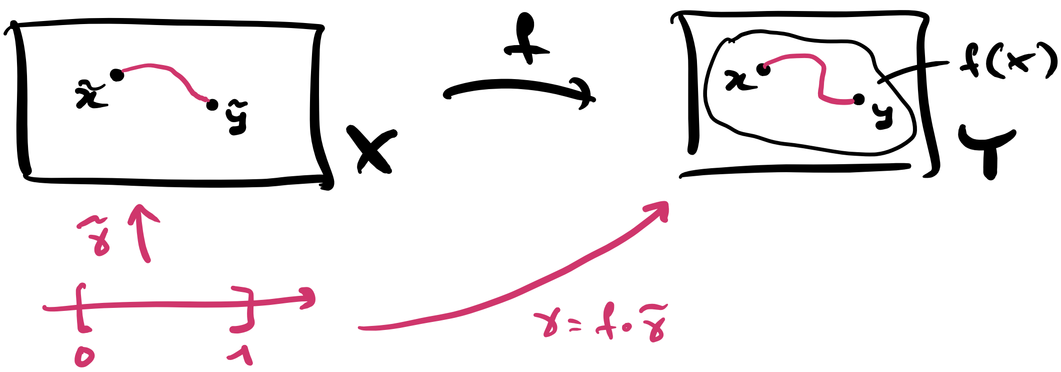
a path from x to y

Prop. Let $f : X \to Y$ be a cont. map. Then

$$[X \text{ pathconnected}] \Longrightarrow [f(X) \text{ pathconnected}].$$

Pf. Let $x, y \in f(X)$. Then there exists $\tilde{x}, \tilde{y} \in X$ s.t. $x = f(\tilde{x})$, $y = f(\tilde{y})$.
By pathconnectedness of X, there is a path $\tilde{\gamma} : [0,1] \to X$ from $\tilde{x}$ to $\tilde{y}$.
Then $\gamma = f \circ \tilde{\gamma} : [0,1] \to f(X)$ is a path from x to y. $\square$

Picture:



Prop. Let $(X, \mathcal{T})$ be a top sp. Then

$$[X \text{ pathconnected}] \Longrightarrow [X \text{ connected}].$$

Pf. Suppose for a contradiction that X is pathconnected but not connected, i.e. $\exists\, U, V \in \mathcal{T}$ such that

$$U \neq \emptyset, \quad V \neq \emptyset, \qquad U \cap V = \emptyset, \qquad U \cup V = X.$$

By nonemptyness, we can find $x \in U$, $y \in V$.
By pathconnectedness, we can find cont. map

$$\gamma : [0,1] \to X \quad \text{with} \quad \gamma(0) = x, \; \gamma(1) = y.$$

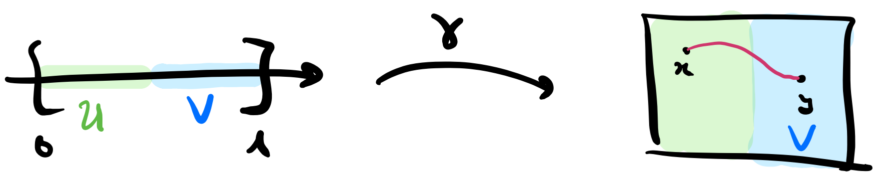
Then $[0,1] = \gamma^{-1}(X) = \gamma^{-1}(U \cup V) = \gamma^{-1}(U) \cup \gamma^{-1}(V)$,

$\gamma^{-1}(U) \cap \gamma^{-1}(V) = \gamma^{-1}(U \cap V) = \gamma^{-1}(\emptyset) = \emptyset$,

$\gamma^{-1}(U) \neq \emptyset$ b/c $\gamma(0) = x \in U$, $\quad \gamma^{-1}(V) \neq \emptyset$ b/c $\gamma(1) = y \in V$,

which contradicts the connectedness of $[0,1]$. ↯ □

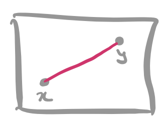
Picture:



Think-pair-share: Pathconnected or not?

$X = \mathbb{R}^n$, $\mathcal{T} = \mathcal{T}_{standard}$

Path connected: $\gamma : [0,1] \to X$, $\gamma(t) = (1-t)x + ty$



$X = \mathbb{R}^n$, $\mathcal{T} = \mathcal{T}_{cofinite}$

Easier for a map to be continuous wrt a coarser topology on the codomain!

Also pathconnected: $\mathcal{T}_{cofinite} \subseteq \mathcal{T}_{standard}$

$X = \mathbb{R}^n$, $\mathcal{T} = \mathcal{T}_{\text{discrete}}$

Not pathconnected b/c not connected.

$X = \mathbb{R} \setminus \{0\}$, $\mathcal{T}$ = subspace top on $\mathbb{R}$.

Not pathconnected. Same reason.

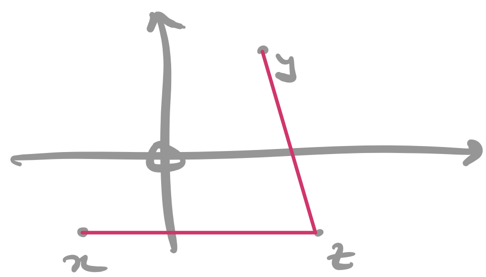
$X = \mathbb{R}^n \setminus \{0\}$ for $n > 1$, $\mathcal{T}$ = subspace topology on $\mathbb{R}^n$

Pathconnected: Let $x, y \in X$.

If 0 is on the line segment between x and y, take a detour:

$\gamma : [0,1] \to X$

$$\gamma(t) = \begin{cases} (1-2t)x + 2t z & \text{for } t \in [0, \frac{1}{2}] \\ (2-2t)z + (2t-1)y & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$



Remark. We can use pathconnectedness to show $\mathbb{R} \not\cong \mathbb{R}^n$ for $n > 1$.

⚠ Not immediately obvious: look up "space-filling curves"!

If there were a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}^n$,
then the restriction $\tilde{\varphi} = \varphi|_{\mathbb{R} \setminus \{0\}} : \mathbb{R} \setminus \{0\} \to \mathbb{R}^n \setminus \{\varphi(0)\}$
would be a homeomorphism, but homeomorphisms preserve pathconnectedness! $\lightning$

More generally: $\mathbb{R}^m \not\cong \mathbb{R}^n$ for $m \neq n$ (see AlgTop 1).

§ The topologist's sine curve

← Example of a connected but not pathconnected space!

117 in the (fantastic!) book "Counterexamples in topology"

$$A := \left\{ \left(x, \sin\left(\tfrac{1}{x}\right)\right) : x \in (0,1] \right\} \subseteq \mathbb{R}^2$$

Both pathconnected and connected since it's the image of the cont map
$(0,1] \to \mathbb{R}^2$
$x \mapsto (x, \sin(\tfrac{1}{x}))$

$$X := \overline{A} \subseteq \mathbb{R}^2$$

Note: X is connected b/c connectedness is preserved under closure.

Will turn out: Not pathconnected!

Observation: $X = A \cup \left(\{0\} \times [-1,1] \right)$

Pf. $\boxed{\supseteq}$ Let $y \in [-1,1]$. Note that

$$\left(\left(\frac{1}{\arcsin(y) + 2\pi n}, y \right) \right)_{n=1}^{\infty}$$ is a sequence in A converging to $(0,y)$ in $\mathbb{R}^2$.

$\rightsquigarrow (0,y) \in X = \overline{A}$.

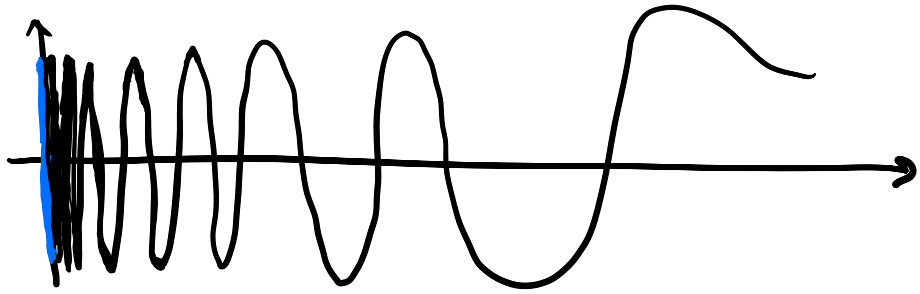
$\boxed{\subseteq}$ Let $(x,y) \in \mathbb{R}^2$ s.t. there exists a sequence

$$\left(\left(x_n, \sin\left(\tfrac{1}{x_n}\right) \right) \right)_{n=1}^{\infty} \longrightarrow (x,y)$$

$x_n \in [0,1)\ \forall n \implies x \in [0,1]$
$\sin(\tfrac{1}{x_n}) \in [-1,1]\ \forall n \implies y \in [-1,1]$

If $x > 0$, then continuity and $x_n \to x$ implies $\sin(\tfrac{1}{x_n}) \to \sin(\tfrac{1}{x})$ but $\sin(\tfrac{1}{x_n}) \to y$ so uniqueness of limits in $\mathbb{R}$ gives $y = \sin(\tfrac{1}{x})$, i.e. $(x,y) \in A$.

Conclude: $(x,y) \in X$. $\square$

Picture: 

Claim: X is <u>not</u> path connected

Pf. Assume for a contradiction that it is.

In particular, $\exists$ path $\gamma : [0,1] \to X$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (\frac{1}{\pi}, 0)$.

Notation: $\gamma_x = \pi_x \circ \gamma : [0,1] \to \mathbb{R}$ x-coordinate of γ
$\gamma_y = \pi_y \circ \gamma : [0,1] \to \mathbb{R}$ y-coordinate of γ
$\}$ continuous maps.

Key idea: γ will have trouble leaving the vertical line segment continuously. We will show this by keeping track of the hilltops and valleys of A.

Note: $\gamma_x(0) = 0$ and $\gamma_x(1) = \frac{1}{\pi}$, so by the intermediate value thm

$$\exists\, t_1 \in (0,1) \quad \text{s.t.} \quad \gamma_x(t_1) = \frac{1}{3\pi/2} \longrightarrow \gamma_y(t_1) = -1$$

Repeated application of the IVT gives

$$\exists\, t_2 \in (0, t_1) \quad \text{s.t.} \quad \gamma_x(t_2) = \frac{1}{5\pi/2} \longrightarrow \gamma_y(t_2) = 1$$

$$\exists\, t_3 \in (0, t_2) \quad \text{s.t.} \quad \gamma_x(t_3) = \frac{1}{7\pi/2} \longrightarrow \gamma_y(t_3) = -1$$

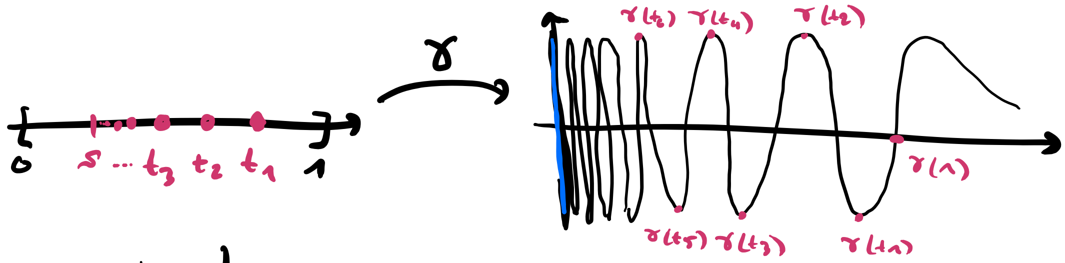
Keep going to get a sequence $t_1 > t_2 > t_3 > t_4 > \cdots > 0$

Such that $\gamma_x(t_n) = \frac{1}{(2n+1)\pi/2}$ and $\gamma_y(t_n) = (-1)^n$.

The sequence $(t_n)_{n=1}^{\infty}$ is decreasing and bounded from below.

Hence, $t_n \to s$ for some $s \in [0,1)$.

Picture:



Now, we're in trouble!

By continuity of γ_y, it should hold that

$$\gamma_y(t_n) \longrightarrow \gamma_y(s)$$

But this is impossible, since

$$\gamma_y(t_n) = (-1)^n \text{ alternates between } \pm 1. \quad \lightning$$

<u>Conclusion:</u> X is <u>not</u> pathconnected. $\square$

<u>Remark.</u> For many well-behaved spaces such as <u>manifolds</u> (spaces that locally look like $\mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 0}$) connectedness actually implies pathconnectedness, b/c they are <u>locally pathconnected</u>. See §8.3 in the book (optional reading).

<u>Exit ticket:</u> True or false?

- If X is connected, and $\sim$ is an equivalence relation, then $X/\sim$ is also connected. <u>True:</u> $X/\sim = \pi(X)$ for the continuous quotient map $\pi: X \to X/\sim$, $x \mapsto [x]$

  <u>Example:</u> $\mathbb{RP}^n$ (path-)connected for all $n \in \mathbb{Z}_{\geq 0}$.
- X pathconnected $\Rightarrow$ $X/\sim$ pathconnected <u>True</u>: Analogous.
- X disconnected $\Rightarrow$ $X/\sim$ disconnected <u>False:</u> $X = \mathbb{R}\setminus\{0\}$, with trivial equivalence relation $x \sim y$ $\forall x, y \in X$

  X disconnected, but $X/\sim \cong \{*\}$ is connected.

To look forward to in the exercises:

- All spaces can be partitioned into connected components.
- $SO(3) \subseteq \mathbb{R}^{3\times 3}$ is pathconnected.
- $GL_3(\mathbb{R})$ and $O(3)$ have two connected components each!

Hint: Look up Euler's rotation theorem to get a feel for how you can move around inside $SO(3)$