PROBLEM SET FOR WEEK 7

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*A subset of these problems will be discussed in the exercise classes in Week 7. Hand in written solutions to **one** of the problems on Absalon by Thursday, March 23 at 18:00.*

Problem 1 (Dimension theory)**.**

(a) Determine the Krull dimension of the following rings (a short motivation is enough):

$$\mathbb{C}[[x]]\,, \quad \mathbb{C}[x^{\pm}]\,, \quad \mathbb{C}[x,y]/(x^2-y, x^3y^2)\,, \quad \mathbb{C}[x,y,z]/(xz,yz)\,, \quad \mathbb{Z}[x]\,.$$

(b) What can we say about the Krull dimension of a PID?

(c) Prove that $\dim(R\times S) = \max\{\dim(R), \dim(S)\}$ for rings R and S.

(d) Nagata's example shows that being Noetherian is not a sufficient condition for a ring to be finite-dimensional. Prove that it is also not a necessary condition.

Problem 2 (More on algebraic sets)**.**

(a) Combine relevant results from previous problem sets with Hilbert's Nullstellensatz to prove that if k is an algebraically closed field, then $\mathbb{V}(-)$ and $\mathbb{I}(-)$ give bijective correspondances

$$\begin{array}{ccc}
\{\text{Algebraic sets } X \subseteq k^n\} & \longleftrightarrow & \{\text{Radical ideals of } k[x_1,\dots,x_n]\} \\
\uparrow & & \uparrow \\
\{\text{Irreducible algebraic sets } X \subseteq k^n\} & \longleftrightarrow & \{\text{Prime ideals of } k[x_1,\dots,x_n]\} \\
\uparrow & & \uparrow \\
\{\text{Singletons } X \subseteq k^n\} & \longleftrightarrow & \{\text{Maximal ideals of } k[x_1,\dots,x_n]\}\,.
\end{array}$$

Give some examples that show how this can fail when k is not algebraically closed.

(b) Let $X \subseteq k^n$ be an algebraic set for an algebraically closed field k. Prove that there is a bijective correspondence between X and maximal ideals of the coordinate ring $k[x_1,\dots,x_n]/\,\mathbb{I}(X)$.

(c) Let k be algebraically closed, let $m < n$, and consider the composition

$$f\colon k[x_1,\dots,x_m] \hookrightarrow k[x_1,\dots,x_n] \to k[x_1,\dots,x_n]/\,\mathbb{I}(X)\,, \quad p \mapsto [p]\,.$$

What does $\operatorname{Spec}(f)$ do to the maximal ideals of $k[x_1,\dots,x_n]/\,\mathbb{I}(X)$? In the light of part (b), interpret $\operatorname{Spec}(f)$ restricted to the maximal ideals as a map $X \to k^m$. Prove that if f is finite and injective, then the corresponding map $X \to k^m$ is surjective, with finite fibers.

Problem 3 (Noether normalization)**.** Recall that a ***Noether normalization*** of a k-algebra R is an injective, finite k-algebra homomorphism of the form $\varphi\colon k[t_1,\dots,t_d] \to R$ for $d \in \mathbb{N}$.

(a) What does the existence of a Noether normalization of a k-algebra R tell you about $\dim(R)$?

(b) Find explicit Noether normalizations of the following $\mathbb{C}$-algebras:

$$\mathbb{C}[x,y]/(xy-1)\,, \quad \mathbb{C}[x,y]/(xy)\,, \quad \mathbb{C}[x,y,z]/(xy+yz+xz)\,.$$

(c) Prove that the automorphism in the proof of the Noether normalization lemma can be chosen to be linear if the field is infinite, by following the steps in Exercise 15.7.

(d) Prove that any algebraic variety $X \subseteq \mathbb{C}^n$ with infinitely many points is unbounded with respect to the Euclidean metric. Is this true over $\mathbb{R}$?

Problem 4 (Nakayama and Cayley–Hamilton)**.**

(a) Let R be a ring. Prove that for a finitely generated R-module M, any surjection $f\colon M \to M$ is an isomorphism. *Hint:* View M as an $R[x]$-module with $x.m = f(m)$ and consider the ideal $I = (x) \subseteq R[x]$.

(b) Prove that if M and N are finitely generated R-modules, such that $M \cong M \oplus N$, then $N \cong 0$. *Hint:* Use part (a). Is the statement true without the assumption that M and N are finitely generated?

(c) Let R be a ring. Prove that there cannot exist an R-linear injection $R^m \to R^n$ if $m > n$. *Hint:* Use Cayley–Hamilton.

(d) Prove that if M is a finitely generated R-module with genering set $\{x_1, \dots, x_n\}$, then any linearly independent subset of M has at most n elements. *Hint:* Use part (c).