PROBLEM SET FOR WEEK 5

OSKAR HENRIKSSON

These problems will be discussed in the exercise classes in Week 5. Hand in clear, independently written solutions to ***one*** *of the problems on Absalon by Thursday, March 9 at 18:00.*

Problem 1 (More on Artinian rings)**.**

(a) Recall that if R is Artinian, then $\operatorname{Spec}(R)$ is finite with the discrete topology. Prove that the converse is not true. *Hint:* Consider $R = k[x_1, x_2, \ldots]/(x_1, x_2^2, x_3^3, \ldots)$.

(b) Let R_1 and R_2 be rings, let S be a local ring, and let $\varphi\colon R_1 \times R_2 \to S$ be a ring homomorphism. Prove that either $\varphi|_{R_1 \times 0} = 0$ or $\varphi|_{0 \times R_2} = 0$.

(c) Let $R_1, \ldots, R_m$ and $S_1, \ldots, S_n$ be local rings such that $\prod_{i=1}^m R_i \cong \prod_{j=1}^n S_j$. Prove that $m = n$ and that there exists a permutaion $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $R_{\sigma(i)} \cong S_i$ for each $i \in \{1, \ldots, n\}$. Use this to conclude that the decomposition in the structure theorem of Artinian rings is unique. *Hint:* Use part (b).

Problem 2 (Tensor products)**.**

(a) Which more familiar modules are the following tensor products isomorphic to:

$$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}\,, \quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}\,, \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}\,, \quad \mathbb{Q}[x] \otimes_{\mathbb{Q}} \mathbb{C}\,?$$

(b) Let M, N and L be modules over a ring R. Prove that $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$ as R-modules. *Hint:* See Exercise 10.15 in the notes.

(c) Prove that $R/I \otimes_R R/J \cong R/(I + J)$ as R-algebras. Give an example for $R = \mathbb{C}[x, y]$.

(d) Look up the definition of a *natural isomorphism* between two functors. Let R be a ring, and let $T \subseteq R$ be a multiplicative subset. Prove that the two functors $R_T \otimes_R -$ and $-_T$ from R**Mod** to R_T**Mod** are naturally isomorphic.

(e) Look up the definition of a *coproduct* in a category. What is the coproduct of two objects in the categories R**Mod**, R**Alg** and **Ring**, respectively? Is there a coproduct of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ in **Field**?

Problem 3 (Flatness)**.**

(a) Prove that the tensor product of flat R-modules is flat, and that the tensor product of faithfully flat R-modules is faithfully flat.

(b) Let R be a ring with a maximal ideal $\mathfrak{m} \subsetneq R$. Prove that $R_{\mathfrak{m}}$ is flat. Also prove that $R_{\mathfrak{m}}$ is faithfully flat if and only if $\mathfrak{m}$ is the only maximal ideal.
Hint: See Exercise 11.15 in the notes.

(c) Let $\varphi\colon R \to S$ be a ring homomorphism, making S into an R-algebra. Let M be an R-module. Prove the following:

(i) if M is flat over R, then $\varphi^* M = S \otimes_R M$ is flat over S;

(ii) if M is faithfully flat over R, then $\varphi^* M = S \otimes_R M$ is faithfully flat over S.

Hint: See Exercise 12.11 in the notes.