PROBLEM SET FOR WEEK 4

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These problems will be discussed in the exercise classes in Week 4. Hand in clear, independently written solutions to ***one*** *of the problems on Absalon by Thursday, March 2 at 18:00.*

Problem 1 (Noetherian and Artinian rings)**.**

(a) Which of the following rings are Noetherian/Artinian/local? (Briefly comment on why, but you do not need to give detailed proofs.) Illustrate your conclusions with a Venn diagram!

$$\mathbb{C}[[x]]\,,\ \mathbb{C}[x^{\pm}]\,,\ \mathbb{C}(x)\,,\ \left\{\tfrac{f}{g} \in \mathbb{C}(x) : g(e^{it}) \neq 0 \text{ for all } t \in \mathbb{R}\right\},\ \mathbb{C}[x]/(x^2 - x)\,,\ \mathscr{C}([0,1])$$

(b) Find an example of a non-Noetherian local ring.

(c) Show that being Noetherian is *not* a local property by finding a non-Noetherian ring R for which $R_{\mathfrak{p}}$ is Noetherian for all $\mathfrak{p} \in \operatorname{Spec}(R)$. *Hint:* Take $R = \prod_{i=1}^{\infty} \mathbb{Z}/2$. Note that each $R_{\mathfrak{p}}$ is local, with all elements being idempotent, and prove that this implies that $R_{\mathfrak{p}}$ is a field.

Problem 2 (Hilbert's basis theorem)**.**

(a) Prove that any finitely generated R-algebra is a Noetherian ring, if R is a Noetherian ring.

(b) Prove the following converse to Hilbert's basis theorem:

If R is a ring such that $R[x]$ is Noetherian, then R is also Noetherian.

(c) Prove that a ring R is Noetherian if and only if $R[[x]]$ is Noetherian. *Hint:* Use the same strategy as in the proof of Hilbert's basis theorem, but focus on lowest-degree terms instead.

Problem 3 (More on algebraic sets)**.** Let k be a field.

(a) Show that any set $S \subseteq k[x_1, \ldots, x_n]$ has a finite subset $S' \subseteq S$ such that $\mathbb{V}(S') = \mathbb{V}(S)$.

(b) Prove that $\mathbb{V}(\mathbb{I}(X)) = X$ for any algebraic set $X \subseteq k^n$. Conclude that $\mathbb{I}(-)$ gives an injection

$$\{\text{Algebraic sets } X \subseteq k^n\} \longrightarrow \{\text{Radical ideals of } k[x_1, \ldots, x_n]\}\,.$$

(c) What does $k[x_1, \ldots, x_n]$ being Noetherian tell us about chains of algebraic sets in k^n?

(d) Let $X \subseteq k^n$ be an algebraic set. Consider the coordinate ring

$$\mathbb{A}(X) = \{\text{Polynomial functions } X \to k\} \cong k[x_1, \ldots, x_n]/\,\mathbb{I}(X)\,.$$

Prove that $\mathbb{A}(X)$ is Artinian if and only if X is a finite set.

Problem 4 (Primary decomposition)**.**

(a) Give two different minimal primary decompositions of the ideal $I = (x^2yz^2, y^2z^2) \subseteq \mathbb{C}[x, y, z]$. What are the associated primes of I? Which ones are isolated/embedded?

(b) Let k be a field. An algebraic set $X \subseteq k^n$ is called ***irreducible*** if it is nonempty and cannot be written as a union of smaller algebraic sets. Later in the course, the Nullstellensatz will tell us that if k is algebraically closed, then $\mathbb{I}(-)$ and $\mathbb{V}(-)$ give a bijective correspondence

$$\{\text{Irreducible algebraic sets } X \subseteq k^n\} \longleftrightarrow \{\text{Prime ideals of } k[x_1, \ldots, x_n]\}\,.$$

Show that this fails when $k = \mathbb{R}$. *Hint:* Consider the ideal $\mathfrak{p} = ((x^2 - 1)^2 + y^2) \subseteq \mathbb{R}[x, y]$.

(c) Prove the ***second uniqueness theorem*** for primary decomposition: Let R be Noetherian, $I \subseteq R$ an ideal, and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ a primary decomposition. Suppose that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i} \in \operatorname{Ass}'(I)$ form some $i \in \{1, \ldots, n\}$. Then $\mathfrak{q}_i$ is given by $\mathfrak{q}_i = \bigcup_{x \notin \mathfrak{p}_i} (I : x)$, and is hence independent of the choice of minimal primary decomposition of I. Compare with part (a).

(d) Find an example of a non-Noetherian ring with an ideal that is not an intersection of finitely many primary ideals. *Hint:* Consider $R = \mathscr{C}([0,1])$ and $I = (0)$. Recall what the maximal ideals of R are from Problem 4, Week 2. Show that any primary ideal of R is contained in a unique maximal ideal, and show that any finite intersection of primary ideals is nonzero.