PROBLEM SET FOR MONDAY WEEK 1

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This first week of the course, we will discuss smaller warm-up problems to help you get used to thinking about rings, modules and algebraic sets. You don't need to hand in anything this week.

Convention: Unless stated otherwise, all rings will be assumed to be commutative and unital throughout the exercise sessions.

Problem 1 (Rings)**.**

(a) What are the initial and terminal objects in the category of rings?

$\mathbb{Z}$ and 0 (up to isomorphism)

(b) Find rings R and S, and a nonzero map $\varphi\colon R \to S$ such that $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for all $a, b \in R$, but for which $\varphi(1_R) \neq 1_S$.

For example, the map $\mathbb{Z} \to \mathbb{Z}/6$ given by $n \mapsto [3n]$, or the map $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $n \mapsto (n, 0)$.

(c) Describe the set $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}/m, \mathbb{Z}/n)$ for all possible choices of $n, m \in \mathbb{Z}_{\geq 0}$.

It's empty if $n \nmid m$, and a singleton if $n \mid m$.

Since a ring homomorphism needs to send the multiplicative identity to the multiplicative identity, the only possible ring homomorphism $\mathbb{Z}/m \to \mathbb{Z}/n$ is $[x]_m \mapsto [x]_n$. This is a well-defined ring homomorphism precisely when $n \mid m$.

Note: In the category of abelian groups, we instead have that $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/m, \mathbb{Z}/n)$ has $\gcd(m, n)$ elements.

(d) Let R be a ring. Prove that there is a bijective correspondance $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \leftrightarrow R$.

Consider the map $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \to R$ with $(\varphi\colon \mathbb{Z}[x] \to R) \mapsto \varphi(x)$ and prove that it's a bijection.

(e) Prove that the abelian group $(\mathbb{Q}/\mathbb{Z}, +)$ does not admit a ring structure.

Assume for a contradiction that there is a multiplication operation $\circ$ and a multiplicative identity e that turns $(\mathbb{Q}/\mathbb{Z}, +)$ into a ring. If $e = [m/n]$ for some $n \in \mathbb{Z}$, then

$$\underbrace{e + \cdots + e}_{n \text{ times}} = [m] = [0]\,.$$

Use the ring axioms to show that this implies that $x + \cdots + x$ (n times) is $[0]$ for all $x \in \mathbb{Q}/\mathbb{Z}$, and derive a contradiction.

(f) Let R and S be rings. Is there a general way to turn $\mathrm{Hom}_{\mathbf{Ring}}(R, S)$ into a ring? (What about using pointwise addition and multiplication?)

No! For instance, $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}/2, \mathbb{Z}/3) = \varnothing$, and it's impossible to put a ring structure on the empty set, since we need to have an additive identity element.

(Compare this to the situation in the category of abelian groups, where one actually *can* equip each hom set $\mathrm{Hom}_{\mathbf{Ab}}(A, B)$ with an abelian group structure, using pointwise addition and multiplication.)

Problem 2 (Algebras and modules)**.**

(a) Let A be a ring. How many $\mathbb{Z}$-algebra structures does A admit?

Precisely one! Remember that $\mathbb{Z}$ is the initial object in **Ring**, so there is a unique ring homomorphism $\mathbb{Z} \to A$.

(b) Find rings R and A, such that A admits more than one R-algebra structure.

Take $R = \mathbb{Z}[x]$ and $A = \mathbb{Z}$, and use Problem 1(d).

(c) What are the initial and terminal objects in the category of R-algebras for a ring R?

Up to isomorphism, the initial object is (R, id_R) and the terminal object is $(0, R \to 0)$ (where $R \to 0$ is the constant zero map).

(d) Look up the definition of an R-module. Let R be a ring, and let (A, φ) be an R-algebra. Prove that we get an R-module structure on A by defining the R-action $R \times A \to A$ to be $r.a = \varphi(r) \cdot a$ for $r \in R$ and $a \in A$.

We need to check that $r.(a + b) = r.a + r.b$, $(r + s).a = r.a + s.a$, $(rs).a = r.(s.a)$ and $1_R.a = a$ for all $r, s \in R$ and $a, b \in A$. For instance, we have

$$r.(a + b) = \varphi(r)(a + b) = \varphi(r)a + \varphi(r)b = r.a + r.b\,.$$

The other properties are shown similarly.

(e) An R-algebra is said to be of ***finite type*** if it is finitely generated as an R-algebra, and is said to be ***finite*** if it is finitely generated when viewed as an R-module. Spell out what this means. Prove that all finite R-algebras are of finite type. Is the converse true?

If an R-algebra A is finite, then there exists an $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$ such that for any $b \in A$, there exists $r_1, \ldots, r_n \in R$ such that

$$b = r_1.a_1 + \cdots + r_n.a_n = \varphi(r_1)a_1 + \cdots + \varphi(r_n)a_n\,.$$

In particular, this gives that it's of finite type.

The converse is not true! Consider for example the case with $R = \mathbb{C}$ and $A = \mathbb{C}[x]$, with $\varphi\colon \mathbb{C} \hookrightarrow \mathbb{C}[x]$ being the usual inclusion. Then $\mathbb{C}[x]$ is of finite type (it's generated by x), but it's an infinite-dimensional $\mathbb{C}$-vector space, so it's not a finite $\mathbb{C}$-algebra.