SUPPLEMENTAL NOTES ON THE EXTENSION THEOREM

In many cases, the following reformulation of the extension theorem is a bit simpler to use in practice than the one given in the course literature and lecture notes.

Theorem 0.1 (The extension theorem, reformulated). *Let $I \subseteq \mathbb{C}[t_1, \dots, t_n]$ be an ideal, let $I_1 = I \cap \mathbb{C}[t_2, \dots, t_n]$, and let $(a_2, \dots, a_n) \in \mathbb{V}(I_1)$. Suppose there exists a polynomial $f \in I$ of the form*

$$f = c(t_2, \dots, t_n)t_1^N + [\textit{terms with lower degree in } t_1],$$

such that $c(a_2, \dots, a_n) \neq 0$. Then there exists an $a_1 \in \mathbb{C}$ such that $(a_1, a_n, \dots, a_n) \in \mathbb{V}(I)$.

Exercise: Prove this is equivalent to the version of the extension theorem stated in the notes.

This has two important consequences that we will use in the exercise sessions. Make sure you understand how they follow from the extension theorem.

Corollary 0.2. *Let $I \subseteq \mathbb{C}[t_1, \dots, t_n]$ be an ideal, and let $I_1 = I \cap \mathbb{C}[t_2, \dots, t_n]$. Suppose there exists a polynomial $f \in I$ of the form*

$$f = c(t_2, \dots, t_n)t_1^N + [\textit{terms with lower degree in } t_1]$$

*such that $c(t_2, \dots, t_n)$ is a nonzero constant. Then for **any** point $(a_2, \dots, a_n) \in \mathbb{V}(I_1)$ there exists an $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \dots, a_n) \in \mathbb{V}(I)$. In other words, the projection $\pi \colon \mathbb{V}(I) \to \mathbb{V}(I_1)$ is surjective.*

Corollary 0.3 (The "free extension lemma"). *Let $I \subseteq \mathbb{C}[t_1, \dots, t_n]$ be an ideal, and let $I_1 = I \cap \mathbb{C}[t_2, \dots, t_n]$. Suppose that I admits a generating set only involving the variables $t_2, \dots, t_n$. Then for any point $(a_2, \dots, a_n) \in \mathbb{V}(I_1)$ and **any** $a_1 \in \mathbb{C}$, it holds that $(a_1, a_2, \dots, a_n) \in \mathbb{V}(I)$. In other words, $\mathbb{V}(I) = \mathbb{C} \times \mathbb{V}(I_1)$. In particular, the projection $\pi \colon \mathbb{V}(I) \to \mathbb{V}(I_1)$ is surjective.*

The results above are formulated in the setting where one has eliminated only one variable, but by extending one variable at a time, one can apply these techniques also when one has eliminated any number of variables. Let us now make this precise.

Suppose that we have an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$ with elimination ideal $I_m = I \cap \mathbb{C}[y_1, \dots, y_n]$, and want to decide whether a certain point $(b_1, \dots, b_n) \in \mathbb{V}(I_m)$ can be extended, in the sense that there exists some $a_1, \dots, a_m \in \mathbb{C}$ such that $(a_1, \dots, a_m, b_1, \dots, b_n) \in \mathbb{V}(I)$.

We will apply the tools described above step-wise, by considering the following sequence of intermediate elimination ideals:

- $I_1 = I \cap \mathbb{C}[x_2, \dots, x_m, y_1, \dots, y_n]$
- $I_2 = I \cap \mathbb{C}[x_3, \dots, x_m, y_1, \dots, y_n]$
- $I_3 = I \cap \mathbb{C}[x_4, \dots, x_m, y_1, \dots, y_n]$
- ...
- $I_{m-2} = I \cap \mathbb{C}[x_{m-1}, x_m, y_1, \dots, y_n]$
- $I_{m-1} = I \cap \mathbb{C}[x_m, y_1, \dots, y_n]$
- $I_m = I \cap \mathbb{C}[y_1, \dots, y_n]$

and the projections

$$\mathbb{V}(I) \longrightarrow \mathbb{V}(I_1) \longrightarrow \mathbb{V}(I_2) \longrightarrow \mathbb{V}(I_3) \longrightarrow \cdots \longrightarrow \mathbb{V}(I_{m-1}) \longrightarrow \mathbb{V}(I_m)$$

$$(x_1, x_2, x_3, \dots, x_m, y_1, \dots, y_n) \mapsto (x_2, x_3 \dots, x_m, y_1, \dots, y_n) \mapsto \cdots \mapsto (x_m, y_1, \dots, y_n) \mapsto (y_1, \dots, y_n)\,.$$

To prove that $(b_1, \dots, b_n)$ can be extended, the idea is to start from the end of this sequence, and step-by-step prove that we can find a sequence of coordinates $a_m, a_{m-1}, \dots, a_2, a_1 \in \mathbb{C}$ such that $(a_1, \dots, a_m, b_1, \dots, b_n) \in \mathbb{V}(I)$.

Application: Proving surjectivity. Suppose we have a polynomial map

$$F = (f_1, \dots, f_n)\colon \mathbb{C}^m \to \mathbb{C}^n,$$

and want to investigate whether it is surjective. We can do this by combining the implicitization and extension theorems, in the following three main steps:

(1) Check what the smallest variety containing the image is.

(2) Try to lift points in this variety with extension.

(3) Analyze special cases where the extension fails by repeating step 1 and 2.

The first step is to form the ideal

$$I := \Big\langle y_1 - f_1(x_1, \dots, x_m), \dots, y_n - f_1(x_1, \dots, x_m) \Big\rangle \subseteq \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$$

which has the property that $\mathbb{V}(I) \subseteq \mathbb{C}^m \times \mathbb{C}^n$ is the *graph* of F.

The image of F is then the image $\pi_m(\mathbb{V}(I)) \subseteq \mathbb{C}^n$ under the canonical projection $\pi_m\colon \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n$. The *implicitization theorem* is telling us that the smallest variety containing $\pi_m(\mathbb{V}(I))$ is $\mathbb{V}(I_m)$, where $I_m = I \cap \mathbb{C}[y_1, \dots, y_n]$ is the m-th elimination ideal.

By the *elimination theorem*, we can compute a Gröbner basis G_m for I_m by computing a Gröbner basis G of I with respect to the lexicographic ordering $x_1 > \dots > x_m > y_1 > \cdots y_m$, and then setting $G_m = G \cap \mathbb{C}[y_1, \dots, y_n]$. This gives rise to the following condition:

Necessary condition for surjectivity: If $F\colon \mathbb{C}^m \to \mathbb{C}^n$ is surjective, then $I_m = \langle 0 \rangle$.

If this condition is *not* satisfied, then we can immediately rule out surjectivity.

Note that this condition is not sufficient for surjectivity; a simple counterexample is given by $F(x_1, x_2) = (x_1, x_1 x_2)$ (make sure you understand this!). Therefore, if the condition is satisfied, we move on, and try to prove surjectivity by the *extension theorem*. More specifically, we try to apply the tools outlined in the first part of this document to an arbitrary fixed point $(b_1, \dots, b_n) \in \mathbb{C}^n$. If this succeeds, we conclude that F is surjective.

Note: It might happen along the way that we don't find any polynomials that satisfy the assumption in the extension theorem or our "free extension lemma". It's important to note that this does **not** necessarily imply that F is non-surjective.

Instead, we should investigate more closely the points $(b_1, \dots, b_n)$ for which the extension criteria fail, and try to manually prove that they either can or cannot be extended. (We can also try to change the monomial ordering in the hope of making it easier to apply the extension theorem.)