Week 5. List 1: Homotopy continuation and certification

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Exercise 1. The following system describes the intersection of an ellipse and an elliptic curve:

$$\begin{aligned} 2x^2 + y^2 - 4 &= 0 \\ x^3 - 3x - y^2 + 3 &= 0\,. \end{aligned} \tag{1}$$

(a) Verify that the system has finitely many complex solutions and compute how many there are. Argue by finding the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[x, y]/I$, where I is the ideal generated by the two polynomials in (1). Compare your conclusion with the Bézout bound.

(b) Write down the straight line homotopy (with the γ-trick) between the system (1) and the total degree start system, as well as the corresponding Davidenko differential equation.

(c) Use `HomotopyContinuation.jl` to solve the system (1).

(d) Use certification to determine how many real solutions you have found approximations of, and how many of these are positive.

(e) Does the above constitute a *proof* that (1) has precisely one positive solution?

Exercise 2 (Certification)**.** Revisit Exercise 2 on List 1 from Week 3, and certify the solutions you found to the system

$$\begin{aligned} x^2 + y^2 + z^2 - 1 &= 0 \\ x^2 + y^2 + z^2 - 2x &= 0 \\ 2x - 3y - z &= 0\,. \end{aligned}$$

Exercise 3 (The Euclidean distance problem)**.** Let $\mathbf{V}(f) \subseteq \mathbb{R}^2$ be a smooth curve, and let $u \in \mathbb{R}^2 \setminus \mathbf{V}(f)$ be some point outside the curve. The goal of this exercise will be to devise an algebraic approach to determining which $x \in \mathbf{V}(f)$ is the closest to u, i.e., what the *global minimum* of the optimization problem

$$\min_{x \in \mathbf{V}(f)} \|x - u\|^2$$

is. Recall that the *critical points* of this optimization problem are the solutions to the polynomial system

$$\begin{aligned} f(x) &= 0 \\ \nabla f(x) - \lambda(x - u) &= 0\,, \end{aligned} \tag{2}$$

where $\lambda \neq 0$ is a new auxiliary variable, which is introduced (often referred to as a *Lagrange multiplier*) to encode that $x - u$ is parallel to ∇f (i.e., orthogonal to the tangent space $T_x\mathbf{V}(f)$).

We consider the optimization problem when the curve is defined by the quartic

$$f(x_1, x_2) = x_1^2 x_2^2 - 3x_1^2 - 3x_2^2 + 5\,,$$

and the point outside the curve is $u = (1, 2)$.

(a) Find the critical points by solving (2) with homotopy continuation.

(b) Evaluate $\|x - u\|^2$ at the real critical points, and determine which one gives the lowest value.

(c) Plot $\mathbf{V}(f)$ together with u and the real critical points you found (e.g., using GeoGebra).

(d) Discuss to what extent we can *prove* that the value found in (b) is the global minimum.

Exercise 4 (27 lines on a cubic)**.** A classical result from algebraic geometry states that every smooth cubic surface in $\mathbb{C}^3$ contains exactly 27 lines. A particularly nice example, where all the 27 lines are real, is the *Clebsch surface*, which is the variety $\mathbf{V}(f) \subseteq \mathbb{C}^3$ given by

$$f(x,y,z) = 81(x^3+y^3+z^3) - 189(x^2y+x^2z+y^2x+y^2z+xz^2+yz^2) + 54\,xyz \\ + 126(xy+xz+yz) - 9(x^2+y^2+z^2) - 9(x+y+z) + 1\,.$$

The goal of this exercise is to explicitly find these 27 real lines.

Setting up the system: A complex line L in $\mathbb{C}^3$ can be parametrized by

$$\mathbb{C} \to L, \qquad t \mapsto p + tv,$$

where $p = (p_1, p_2, p_3) \in L$ is an arbitrary but fixed point on the line and $v = (v_1, v_2, v_3) \in \mathbb{C}^3 \setminus \{0\}$ gives the direction. Note that the choices of p and v are not unique. However, if we pick a random affine hyperplane in $\mathbb{C}^3$, it will intersect each of the 27 lines precisely once with probability one. For instance, you can try this hyperplane:

$$7 + p_1 + 3p_2 + 5p_3 = 0\,. \tag{3}$$

Similarly, if we impose a random affine relation in v_1, v_2 and v_3, there will be unique v satisfying this for each of the 27 lines with probability one. For instance, you can try

$$11 + 3v_1 + 5v_2 + 7v_3 = 0\,. \tag{4}$$

The line parametrized by $p + tv$ is contained in the surface $\mathbf{V}(f)$ if and only if

$$f(p+tv) = f(p_1+tv_1, p_2+tv_2, p_3+tv_3) = 0 \quad \text{for all } t \in \mathbb{C}\,.$$

Note that $f(p+tv)$ can be viewed as a polynomial in t with coefficients $c_0, c_1, c_2, c_3 \in \mathbb{C}[p,v]$:

$$f(p+tv) = c_0(p,v)t^3 + c_1(p,v)t^2 + c_2(p,v)t + c_3(p,v)$$

and that $f(p+tv)$ vanishes for all $t \in \mathbb{C}$ if and only if

$$c_0(p,v) = 0\,, \quad c_1(p,v) = 0\,, \quad c_2(p,v) = 0\,, \quad c_3(p,v) = 0\,. \tag{5}$$

Thus, to find the 27 lines contained in the Clebsch surface, we should find all pairs $(p,v) \in \mathbb{C}^3 \times \mathbb{C}^3$ satisfying the equations (3), (4) and (5).

(a) Use the commands `subs` and `coefficients` in `Julia` to find $c_0, c_1, c_2, c_3 \in \mathbb{C}[p,v]$.

(b) Solve the system consisting of (3), (4) and (5) with homotopy continuation. Use certification to check that 27 distinct and real solutions are found.

(c) Plot the surface together with some of the lines you found (for instance with the free tool `https://www.math3d.org/8Uq3KNQMi`).

(d) Try to solve the system again, but now use Gröbner bases and elimination in `OSCAR` instead.

Exercise 5 (More on the γ-trick)**.** Consider the straight line homotopy (with the γ-trick)

$$H(t,x) = t\,F(x) + \gamma\,(1-t)\,G(x)\,,$$

between the target system

$$F(x) = \begin{bmatrix} x^2 - y^2 - 1 \\ 2x^2 + y^2 - 8 \end{bmatrix}$$

and the corresponding total degree start system $G(x)$. Find a specific $\gamma \in \mathbb{C} \setminus \{0\}$ such that at least one of the paths encounter a singular Jacobian at $t = 1/2$. Note that this corresponds to finding a γ such that there is a solution of the system

$$H\left(\tfrac{1}{2}, x\right) = 0\,, \qquad \det\left(\frac{\partial H}{\partial x}\left(\tfrac{1}{2}, x\right)\right) = 0\,.$$

See what happens if you force `HomtopyContinuation.jl` to use this particular γ by running

```
solve(F, start_system = :total_degree, gamma = ...).
```