SOME NOTES ON BIRCH'S THEOREM

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Let $A \in \mathbb{Z}^{k\times n}$ be an integer matrix, and consider the monomial map

$$\varphi_A\colon (\mathbb{C}^*)^k \longrightarrow \mathbb{C}^n\,, \qquad \boldsymbol{t} = (t_1,\dots,t_k) \longmapsto \boldsymbol{t}^A = \begin{bmatrix} t_1^{a_{11}}\cdots t_k^{a_{k1}} \\ \vdots \\ t_1^{a_{1n}}\cdots t_k^{a_{kn}} \end{bmatrix}.$$

The Zariski closure of the image in $\mathbb{C}^n$ is the toric variety X_A. In these notes we will be interested in the positive part $(X_A)_{>0} = X_A \cap \mathbb{R}^n_{>0} = \varphi_A(\mathbb{R}^k_{>0})$, and the nonnegative part $(X_A)_{\geq 0} = X_A \cap \mathbb{R}^n_{\geq 0}$. We will also consider scaled toric varieties of the form $\boldsymbol{c} \star X_A$ for some scaling vector $\boldsymbol{c} \in \mathbb{R}^n_{>0}$, where $\star$ denotes component-wise multiplication.

The main goal of these notes will be to understand the intersection of the scaled toric variety $(\boldsymbol{c} \star X_A)_{\geq 0}$ and affine spaces of the form $\boldsymbol{v} + \ker(A)$ for $\boldsymbol{v} \in \mathbb{R}^n_{>0}$. In particular, we prove the following result, which has been rediscovered many times in many fields. In statistics, it is known as Birch's theorem, and in reaction network theory, it goes back to the paper [HJ72].

Theorem 0.1 (Birch's theorem)**.** *Let $A \in \mathbb{Z}^{k\times n}$ and $\boldsymbol{c}, \boldsymbol{v} \in \mathbb{R}^n_{>0}$. Then $|(\boldsymbol{c}\star X_A)_{\geq 0} \cap (\boldsymbol{v} + \ker(A))| = 1$.*

Our strategy will be to first prove that there is a unique intersection point in the positive orthant $\mathbb{R}^n_{>0}$, and then prove that there are no additional points in $\partial\mathbb{R}^n_{\geq 0}$. In the positive orthant, we have the following convenient description of the scaled toric variety, which is why sets of this form are often called *log-affine* in the statistical setting.

Lemma 0.2. *Let $A \subseteq \mathbb{Z}^{k\times n}$ be a linear subspace. Then*

$$(\boldsymbol{c}\star X_A)_{>0} = \{\boldsymbol{x} \in \mathbb{R}^n_{>0} : \log(\boldsymbol{x}) - \log(\boldsymbol{c}) \in \operatorname{row}(A)\}\,.$$

Proof. We begin by proving "$\supseteq$". Suppose $A \in \mathbb{R}^{k\times n}$ is such that $\operatorname{row}(A) = V$. If $\log(\boldsymbol{x}/\boldsymbol{c}) \in V$, then there exists $\lambda_1,\dots,\lambda_k$ such that $\log(\boldsymbol{x}/\boldsymbol{c}) = \sum_{i=1}^k \lambda_i \operatorname{row}_i(A)$, i.e. $\log(x_j/c_j) = \sum_{i=1}^k \lambda_i a_{ij}$ for all $j \in [n]$. Exponentiating gives $x_j = c_j \prod_{i=1}^k (e^{\lambda_i})^{a_{ij}}$, and we conclude that $\boldsymbol{x} = \boldsymbol{c}\star\boldsymbol{t}^A$, where $\boldsymbol{t} = (e^{\lambda_1},\dots,e^{\lambda_k}) \in \mathbb{R}^k_{>0}$. To prove "$\supseteq$", we now simply read this argument backwards. ☐

We now prove existence and uniqueness of the intersection point in $\mathbb{R}^n_{>0}$. In the reaction networks literature, this result have appeared in, e.g., [Fei95, Prop. 5.1 and B.1] and [Bor12, Lem. 3.15]. The proof we give here is a reformulation of the proof given by Boros.

Proposition 0.3. *Let $A \in \mathbb{Z}^{k\times n}$, and let $\boldsymbol{c}, \boldsymbol{v} \in \mathbb{R}^n_{>0}$. Then $|(\boldsymbol{c}\star X_A)_{>0} \cap (\boldsymbol{v} + \ker(A))| = 1$.*

Proof. We begin by proving uniqueness of the intersection point between the scaled toric variety and the affine space. Suppose $\boldsymbol{x}, \boldsymbol{y} \in (\boldsymbol{c}\star X_A)_{>0} \cap (\boldsymbol{v} + \operatorname{row}(A))$. Then $\boldsymbol{x}, \boldsymbol{y} \in (\boldsymbol{c}\star X_A)_{>0}$ implies $\log(\boldsymbol{x}) - \log(\boldsymbol{y}) \in \operatorname{row}(A)$ by Lemma 0.2, whereas $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{v} + \operatorname{row}(A)$ implies $\boldsymbol{x} - \boldsymbol{y} \in \operatorname{row}(A)$. This gives

$$0 = \langle \boldsymbol{x} - \boldsymbol{y}, \log(\boldsymbol{x}) - \log(\boldsymbol{y})\rangle = \sum_{i=1}^n (x_i - y_i)(\log(x_i) - \log(y_i))\,.$$

This is a sum of nonnegative terms (each term is the product of two numbers with the same sign) since $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ is a strictly increasing function. Hence, the only way for the sum to be zero, is if all terms are zero, from which we conclude $\boldsymbol{x} = \boldsymbol{y}$.

Next, we prove existence. The trick will be to introduce the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\boldsymbol{\lambda}) = \langle \boldsymbol{c}, \exp(\boldsymbol{\lambda})\rangle - \langle \boldsymbol{v}, \boldsymbol{\lambda}\rangle = \sum_{i=1}^n \underbrace{\Big(c_i e^{\lambda_i} - v_i\lambda_i\Big)}_{=:\, f_i(\lambda_i)}\,,$$

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and the key observation will be that $f|_{\mathrm{row}(A)}\colon \mathrm{row}(A) \to \mathbb{R}$ has a minimum, say $\boldsymbol{\lambda}^* \in \mathrm{row}(A)$.

To see this, note that each term $f_i(\lambda_i)$ is continuous and satisifes $\lim_{\lambda_i \to \pm\infty} f_i(\lambda_i) = \infty$, which by basic calculus implies that $f_i(\lambda_i)$ attains some minimum value M_i. Let $M = \min\{M_1, \ldots, M_n\}$. Furthermore, if we pick $C > 0$ large enough, it will hold for every $i \in [n]$ that $|\lambda_i| > C$ implies $f_i(\lambda_i) > f(\mathbf{0}) - (n-1)M$. Form the compact set $K = \{\boldsymbol{\lambda} \in \mathbb{R}^n : \|\boldsymbol{\lambda}\|_\infty \leq C\}$. The idea now is that we can restrict our search for a minimum to K, since for any $\boldsymbol{\lambda} \in \mathbb{R}^n \setminus K$, there will be some $i_0 \in [n]$ such that $|\lambda_{i_0}| > C$, which gives

$$f(\boldsymbol{\lambda}) = f_{i_0}(\lambda_{i_0}) + \sum_{i \in [n]\setminus\{i_0\}} f_i(\lambda_i) > (f(\mathbf{0}) - (n-1)M) + (n-1)M = f(\mathbf{0}).$$

Since f is continuous and the intersection $K \cap \mathrm{row}(A)$ is compact and nonempty (note that $\mathbf{0} \in K \cap \mathrm{row}(A)$), we get that $f|_{K\cap\mathrm{row}(A)}$ has a minimum, say $\boldsymbol{\lambda}^*$, which will also be a minimum of $f|_{\mathrm{row}(A)}$, since $f(\boldsymbol{\lambda}) > f(\mathbf{0}) \geq f(\boldsymbol{\lambda}^*)$ for all $\boldsymbol{\lambda} \in \mathrm{row}(A) \setminus K$.

Now, set $\boldsymbol{x} = \boldsymbol{c} \star \exp(\boldsymbol{\lambda}^*)$. Taking logarithms then gives $\log(\boldsymbol{x}) - \log(\boldsymbol{c}) = \boldsymbol{\lambda}^* \in \mathrm{row}(A)$, so that $\boldsymbol{x} \in (\boldsymbol{c} \star X_A)_{>0}$ by Lemma 0.2. Moreover, since f is C^1 with $(\nabla f)(\boldsymbol{\lambda}) = \boldsymbol{c} \star \exp(\boldsymbol{\lambda}) - \boldsymbol{v}$, the basic theory of constrained optimization from multivariable calculus gives that $(\nabla f)(\boldsymbol{\lambda}^*) = \boldsymbol{x} - \boldsymbol{v} \in \mathrm{row}(A)$, from which we obtain $\boldsymbol{x} \in \boldsymbol{v} + \mathrm{row}(A)$. ☐

Next, we turn our attention to the nonnegative part of the toric variety, and ask how many additional points there are in the intersection $(\boldsymbol{c} \star X_A)_{\geq 0} \cap (\boldsymbol{v} + \ker(A))$ compared to $(\boldsymbol{c} \star X_A)_{>0} \cap (\boldsymbol{v} + \ker(A))$.

The starting point of our analysis is the following polyhedral classification of the supports of vectors in $(\boldsymbol{c} \star X_A)_{\geq 0}$. We will use the following notations: For a matrix $A \in \mathbb{Z}^{k\times n}$, we let

$$\mathrm{Conv}(A) := \{A\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \Delta_{n-1}\} \subseteq \mathbb{R}^k$$

be the polytope given by convex linear combinations of the columns of A, and

$$\mathrm{Cone}(A) := \{A\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \mathbb{R}^n_{\geq 0}\} \subseteq \mathbb{R}^k$$

be the cone spanned by the columns of A. For an index set $I \subseteq [n]$, we let A_I be the submatrix of A given by the columns with indices in I.

Following [GMS06], we define a *facial set* with respect to a matrix $A \in \mathbb{Z}^{k\times n}$ to be a set $F \subseteq [n]$ for which there exists a vector $v \in \mathbb{R}^n$ such that $v^\top a_i = 0$ for $i \in F$ and $v^\top a_i \geq 0$ for $i \in [n] \setminus F$. Note that if $F \subseteq [n]$ is a facial set, then $\mathrm{Conv}(A_F)$ is a face of the polytope $\mathrm{Conv}(A)$, and if $F \subsetneq [n]$, then $\mathrm{Conv}(A_F)$ is a proper face of $\mathrm{Conv}(A)$.

Lemma 0.4 ([GMS06, Lem. A.2])**.** *Suppose that $\mathbb{1} \in \mathrm{row}(A)$. Then $\{\mathrm{supp}(\boldsymbol{x}) : \boldsymbol{x} \in (X_A)_{\geq 0}\}$ is the set of all facial sets with respect to A.*

Note that if $\boldsymbol{c} \in \mathbb{R}^n_{>0}$, then the supports of elements in $(\boldsymbol{c} \star X_A)_{\geq 0}$ are the same as the supports of elements in $(X_A)_{\geq 0}$. Furthermore, if $\mathbb{1} \notin \mathrm{row}(A)$, then we can just add such a row to form a new matrix $\tilde{A}$, and find the facial sets of this matrix. We note that $X_{\tilde{A}}$ is the union of all lines through the origin and points on X_A, which means that the supports of element in $(X_{\tilde{A}})_{\geq 0}$ are the same same as the supports of elements in $(X_A)_{\geq 0}$.

Proposition 0.5. *Let $A \in \mathbb{Z}^{k\times n}$, and let $\mathbf{c}, \boldsymbol{v} \in \mathbb{R}^n_{>0}$. Then $(\boldsymbol{c} \star X_A)_{\geq 0} \cap (\boldsymbol{v} + \ker(A)) \cap \partial\mathbb{R}^n_{\geq 0} = \emptyset$.*

Proof. Suppose that there exists some $\boldsymbol{x} \in (\boldsymbol{c} \star X_A)_{\geq 0} \cap (\boldsymbol{v} + \ker(A))$ such that $\boldsymbol{x} \notin \mathbb{R}^n_{>0}$, and let $F := \mathrm{supp}(\boldsymbol{x}) \subsetneq [n]$. Then $\boldsymbol{x} \in (\boldsymbol{v} + \ker(A))$ gives that $A\boldsymbol{v} = A\boldsymbol{x}$. But this is impossible since $A\boldsymbol{v}$ lies in the interior of $\mathrm{Cone}(A)$, whereas $A\boldsymbol{x} = A_F\boldsymbol{x}_F$ lies on the proper face $\mathrm{Cone}(A_F)$. ☐

Proof of Theorem 0.1. This follows by Propositions 0.3 and 0.5. ☐

References

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