SUGGESTED SOLUTION TO PROBLEM 14

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This is a suggested solution for Problem 14. If you find something that looks like a typo or error (or if you have questions, or want additional feedback on an attempted solution), feel free to email me. As always, you shouldn't just read through this solution, but actively process it until you're sure that you can write down an own solution, with your owns words.

Preliminaries

Let's begin with clearing up some subtleties that will be important when we solve Problem 14.

In my grading the last couple of weeks, I've really emphasized the importance of always specifying *where* subsets are closed and open, if it's not 100% clear from the context. The issue is the following: If we have a topological space X and a subset $A \subseteq X$ equipped with the subspace topology, then being open/closed in A is not necessarily the same as being open/closed in X.

Example: If $X = \mathbb{R}$ (with the standard topology) and $A = [0, 1]$, then $U = (\frac{1}{2}, 1]$ is open in the subspace topology of A, but it's not open in X.

In particular, this means that you should be very careful about whether your sets are open/closed in X or in X^+ when you solve Problem 14! However, when it comes to compactness, the situation is less tricky, as the following lemma shows.

Lemma: Let X be a topological space, and let $A \subseteq X$ be equipped with the subspace topology. Let $K \subseteq A$. Then K is compact in A if and only if K is compact in X.

Proof. "$\Rightarrow$": Assume that K is compact in A, and let $\{U_i\}_{i\in I}$ be an open covering of K by open subsets of X. Then $\{U_i \cap A\}_{i\in I}$ is an open covering of K by open subsets of A. By assumption, it has a finite subcovering, say $\{U_{i_1} \cap A, \ldots, U_{i_n} \cap A\}$ for some $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$, and then $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcovering of the original covering.

"$\Leftarrow$": Assume that K is compact in X, and let $\{U_i \cap A\}_{i\in I}$ be an arbitrary open covering of K by open subsets of A, where U_i is open in X for every $i \in I$. Then $\{U_i\}_{i\in I}$ is an open covering of K by open subsets of X, which by assumption has a finite subcovering $\{U_{i_1}, \ldots, U_{i_n}\}$ for some $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$. This gives us a finite subcovering $\{U_{i_1} \cap A, \ldots, U_{i_n} \cap A\}$ of the original covering. □

Hence, we don't necessarily have to be as careful about whether subsets of X are compact in X or X^+. (However, I personally think it's a good habit to be precise about this anyways, when one is new to topology.)

Problem and solution

In what follows, X will denote locally compact Hausdorff space, with one-point compactification X^+, and $Y \subseteq X$ will be an arbitrary subset.

Claim 1: Y is closed in X^+ if and only if Y is compact in X.

Proof. "$\Rightarrow$": Note that X^+ is compact, so if Y is closed in X^+ it must be compact in X^+ by Theorem 9.9, which by the lemma gives that Y is compact in X.

"$\Leftarrow$": If Y is compact in X, the lemma gives that it's compact also in X^+. Since X^+ is Hausdorff, this implies that Y is closed in X^+ by Theorem 9.10. □

Claim 2: If Y is closed in X, then $Y \cup \{\infty\}$ is closed in X^+.

Proof. Suppose Y is closed in X. Note that

$$X^+ \setminus (Y \cup \{\infty\}) = (X \cup \{\infty\}) \setminus (Y \cup \{\infty\}) = X \setminus Y\,,$$

which by assumption is open in X, and therefore also open in X^+ by the definition of the one-point compactification. This proves that $Y \cup \{\infty\}$ is closed in X^+. □

Claim 3: Let $\mathrm{Cl}_X(Y)$ be the closure of Y in X, and let $\mathrm{Cl}_{X^+}(Y)$ be the closure of Y in X^+. Then

$$\mathrm{Cl}_{X^+}(Y) = \begin{cases} \mathrm{Cl}_X(Y) & \text{if } \mathrm{Cl}_X(Y) \text{ is compact in } X \\ \mathrm{Cl}_X(Y) \cup \{\infty\} & \text{if } \mathrm{Cl}_X(Y) \text{ is not compact in } X \end{cases}$$

Proof. We begin by proving two inclusions:

"$\mathrm{Cl}_X(Y) \subseteq \mathrm{Cl}_{X^+}(Y)$": Note that $\mathrm{Cl}_{X^+}(Y)$ is a closed subset of X^+ that contains Y. Hence, $\mathrm{Cl}_{X^+}(Y) \cap X$ is a closed in X (recall that the topology of X coincides with the subspace topology in X^+) that contains Y. By definition of the closure, it must then hold that $\mathrm{Cl}_X(Y) \subseteq \mathrm{Cl}_{X^+}(Y) \cap X$, from which it follows that $\mathrm{Cl}_X(Y) \subseteq \mathrm{Cl}_{X^+}(Y)$.

"$\mathrm{Cl}_{X^+}(Y) \subseteq \mathrm{Cl}_X(Y) \cup \{\infty\}$": Note that $\mathrm{Cl}_X(Y)$ is closed in X and contains Y, so by Claim 2, it holds that $\mathrm{Cl}_X(Y) \cup \{\infty\}$ is closed in X^+ and contains Y. Then the definition of the closure gives that $\mathrm{Cl}_{X^+}(Y) \subseteq \mathrm{Cl}_X(Y) \cup \{\infty\}$.

We now have

$$\mathrm{Cl}_X(Y) \subseteq \mathrm{Cl}_{X^+}(Y) \subseteq \mathrm{Cl}_X(Y) \cup \{\infty\}\,.$$

Since the only difference between the left-hand side and the right-hand side is the single point ∞, it must hold that either $\mathrm{Cl}_{X^+}(Y) = \mathrm{Cl}_X(Y)$ or $\mathrm{Cl}_{X^+}(Y) = \mathrm{Cl}_X(Y) \cup \{\infty\}$.

Note that by definition of the closure, $\mathrm{Cl}_{X^+}(Y)$ is the smallest closed subset of X^+ that contains Y. Hence, we're in the case $\mathrm{Cl}_{X^+}(Y) = \mathrm{Cl}_X(Y)$ if and only if $\mathrm{Cl}_X(Y)$ is closed in X^+. By Claim 1, this holds if and only if $\mathrm{Cl}_X(Y)$ is compact in X. The desired result follows. □