SUGGESTED SOLUTION TO PROBLEM 12

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This is a suggested solution for Problem 12. If you find something that looks like a typo or error (or if you have questions, or want additional feedback on an attempted solution), feel free to email me.

Claim: Let (X, d) be a metric space, let $A \subseteq X$ be a closed subset, and let $B \subseteq X$ be a compact subset. Also let $D := \{d(x, y) : x \in A, y \in B\}$. Then it holds that

$$A \cap B \neq \varnothing \iff \inf(D) = 0\,.$$

Proof. "$\Rightarrow$": Suppose that $A \cap B \neq \varnothing$. Then there exists some $z \in A \cap B$, and we get that $0 = d(z, z) \in D$, from which it follows that $\inf(D) \leq 0$. At the same time, the definition of a metric (see Definition 1.1 in the script) gives that all elements in D are nonnegative, so $\inf(D) \geq 0$. Hence, we conclude that $\inf(D) = 0$.

"$\Leftarrow$": Suppose that $\inf(D) = 0$. Then, for any $n \in \mathbb{N}$, we can find $x_n \in A$ and $y_n \in B$ such that $d(x_n, y_n) < 1/n$. This gives a sequence $(y_n)_{n=1}^{\infty}$ of elements in B. Since B (with the restriction of d) is a metric space, compactness implies sequential compactness (see Theorem 10.5 in the script). Hence, there is a subsequence $(y_{n_k})_{k=1}^{\infty}$ (for some $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$) that converges to some $y \in B$.

For each $k \in \mathbb{N}$, the triangle inequality now gives that

$$0 \leq d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y)\,.$$

Since both terms of the right-hand side go to 0 as $k \to \infty$, the "squeezing theorem" from calculus gives that $d(x_{n_k}, y) \to 0$ as $k \to \infty$, and we conclude that $(x_{n_k})_{k=1}^{\infty}$ converges to y. This implies that $y \in \overline{A}$, but since A is closed, $\overline{A} = A$, and we conclude that $y \in A$. Hence, we have $y \in A \cap B$, which shows that $A \cap B \neq \varnothing$. $\square$

Alternative proof of "$\Leftarrow$". We prove this by contraposition. Suppose $A \cap B = \varnothing$. Then, for any $y \in B$, it holds that $y \in X \setminus A$. Note that $X \setminus A$ is open in X, since A is closed. Hence, we can find some $\varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \subseteq X \setminus A$.

Key observation: We have an open covering $\{B_{\varepsilon_y/2}(y) : y \in B\}$ of B, and by compactness of B, we can find a finite subcovering, i.e. there exists some $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in B$ such that

$$B \subseteq \bigcup_{i=1}^{n} B_{\varepsilon_{y_i}/2}(y_i)\,. \tag{1}$$

Let $\varepsilon := \min\{\varepsilon_{y_1}, \ldots, \varepsilon_{y_n}\}$. Since it's the minimum of finitely many positive numbers, we have $\varepsilon > 0$.

Remark: Note that we have divided the radii by 2. This is a trick that we use to get a strictly positive bound in the estimation (2) later in the proof.

Now let $x \in A$ and $y \in B$ be arbitrary. Then (1) gives there exists some $i \in \{1, \ldots, n\}$ such that $y \in B_{\varepsilon_{y_i}/2}(y_i)$. By the triangle inequality, it holds that

$$d(x, y) \geq d(x, y_i) - d(y, y_i) > \varepsilon_{y_i} - \frac{\varepsilon_{y_i}}{2} = \frac{\varepsilon_{y_i}}{2} \geq \frac{\varepsilon}{2}\,, \tag{2}$$

where the strict inequality sign follows from the following two observations:

- By assumption, $B_{\varepsilon_{y_i}}(y_i) \subseteq X \setminus A$, so $x \notin B_{\varepsilon_{y_i}}(y_i)$, which implies that $d(x, y_i) \geq \varepsilon_{y_i}$.
- By assumption, $y \in B_{\varepsilon_{y_i}}(y_i)$, so it holds that $d(y, y_i) < \varepsilon_{y_i}$.

(It's a good idea to draw a picture of this in $\mathbb{R}^2$ to help your intuition!)

Put differently, the estimation in (2) shows that $\varepsilon/2$ is a lower bound of D, from which it follows that $\inf(D) \geq \varepsilon/2 > 0$. In particular, $\inf(D) \neq 0$, which is what we wanted to show! $\square$