HINTS AND COMMENTS ON THE EXERCISES FROM WEEK 5–7

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These are some very quickly compiled comments regarding the last couple of exercise sheets. If you find something that looks like a typo or error (or if you have questions, or want feedback on an attempted solution), feel free to email me at `oskar.henriksson@math.ku.dk`. *The latest version of this document can be found at* `http://oskarhenriksson.se/teaching/topology-hints.pdf`.

Exercises from Week 5

Exercise 17. Use a similar trick as in the proof of Corollary 8.17.

Useful lemma to prove: If $\varphi\colon X \to Y$ is a homeomorphism between topological spaces X and Y, with $x_0 \in X$, then the restriction $\tilde{\varphi}\colon X \setminus \{x_0\} \to Y \setminus \{\varphi(x_0)\}$ is also a homeomorphism.

Exercise 18. Here it's a good idea to introduce some notation for the connected components. For instance, for each $x \in X$, we can let Γ_x be the connected component of x (i.e. $[x]$ if we view it as an equivalence class).

Useful lemma for part (ii): For any $x \in X$, the connected component Γ_x is connected. (Try using Lemma 8.7 in the script to prove this.)

Useful lemma for the final part of the exercise: If X is a connected space, and $\sim$ is an equivalence relation on X, then the quotient space $X/{\sim}$ is also connected. (Use Proposition 8.5 to prove this.)

To show that the space X from Exercise 16 is connected is a hard problem! You could try to prove it directly, by assuming for a contradiction that X is disconnected. A somewhat more smooth and elegant possibility is the following:

- Show that $X_{<0}$ and $X_{>0}$ are both connected.
- Show that $\overline{X_{<0}} = X_{\leq 0}$ and $\overline{X_{>0}} = X_{\geq 0}$.
- Recall that the closure of a connected subset is again connected.
- Note that $X = X_{\leq 0} \cup X_{\geq 0}$, and apply Lemma 8.7.

The intuition behind this proof is that the main difficulty of X (compared to $\mathbb{R}$ with the standard topology) is that neighborhoods of 0 are complicated. So it makes sense to begin the proof by ignoring 0, and instead analyze the positive and negative parts separately. In this way, we can make some easier progress early on, that we can then build on for the rest of the proof.

Exercise 19. The classical trick for solving this problem is to do the following:

Let $x \in U$ be arbitrary, and form the set

$$P = \{y \in U : \text{there is a path } \gamma\colon [0,1] \to U \text{ from } x \text{ to } y\}.$$

Our goal then becomes to prove that $P = U$. *Hint:* Prove that P is clopen in U. Draw pictures in $\mathbb{R}^2$ for intuition.

Exercise 20. The key for both (i) and (ii) is to apply Lemma 8.7 in a clever way.

For (i), you can note that $A \cup \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}(A_i \cup A)$.

For (ii), you can note that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}(\bigcup_{j=1}^{i} A_i)$.

Recommendation: Draw pictures in $\mathbb{R}^2$ to get a feeling for what's going on here!

Exercises from Week 6

Exercise 21. This is the so-called *extreme value theorem*, which you might recognize from your calculus class! (If not, it might be a good idea to quickly go back and review it!)

This appears as Corollary 9.13 in the script, but the proof given there is a bit short, so it's a good idea to try to fill in all the details yourself!

The first time you go through the proof, I recommend that you think of Y as just $\mathbb{R}$ with the usual ordering $<$. (Note that the corresponding order topology on $\mathbb{R}$ coincides with the standard topology.) You can also use the more familiar notation $(-\infty, z)$ and (z, ∞) for $Y_{z,-}$ and $Y_{z,+}$, respectively.

Another piece of advice is to divide the statement into two separate statements:

(i) There exists some $x_+ \in X$ such that $f(x) \leq f(x_+)$ for all $x \in X$ (in other words: the image $f(X)$ has a maximal element, namely $f(x_+)$).

(ii) There exists some $x_- \in X$ such that $f(x_-) \leq f(x)$ for all $x \in X$ (in other words: the image $f(X)$ has a minimal element, namely $f(x_-)$).

Exercise 22. Part (i) is mainly just a matter of checking the definition of compactness. For part (ii), you can use (i), combined with Theorem 9.9 and Theorem 9.10.

Exercise 23. There are two statements to check in this exercise.

The first statement is false. If $f\colon X \to Y$ is a continuous map between topological spaces, and X is limit point compact, then it's *not* necessarily true that the image $f(X) \subseteq Y$ is limit point compact.

As a counterexample, you can use the trick explained in Example 10.3, namely that any space can be forced to become limit point compact by taking the product with $\{0, 1\}$ with the trivial topology.

More specifically, you can let $\mathbb{N}$ be equipped with the discrete topology (which is definitely non-compact, why?), and form the product $X = \mathbb{N} \times \{0, 1\}$ with the product topology. Let $f\colon X \to \mathbb{N}$ be the projection onto the first factor, i.e. the map given by $f(n, i) = n$.

The second statement is true. If X is a limit point compact space, and $Y \subseteq X$ is a closed subspace, then Y is also limit point compact.

Hint: What does Y being closed have to do with limit points?

Exercise 24. The statement this problem asks you to prove is called the *Alexander subbase theorem*. It's genuninely hard to prove, and it's a bit beyond the scope of this course. In particular, this **won't be relevant for the exam**.

However, if you're very interested and up for a challenge, you can try to read the proof given on Wikipedia: `https://en.wikipedia.org/wiki/Subbase#Alexander_subbase_theorem`. The proof uses Zorn's lemma, which you might have seen in some previous course. A more condensed proof (where the application of Zorn's lemma is hidden in the innocent-sounding statement "we may assume") can be found here: `https://people.clas.ufl.edu/kees/files/AlexanderTychonoff.pdf`.

Even if you won't need it for the exam, it's very good to be familiar with Zorn's lemma if you plan to continue study pure mathematics. Good places to learn or review it are Wikipedia: `https://en.wikipedia.org/wiki/Zorn%27s_lemma` and this note: `https://www.math.ucla.edu/~tsmits/115B/115A/Zorn's%20Lemma.pdf`, where it's shown how it can be used to prove that every vector space has a basis. Maybe something you can check out once you're done with the exam!

A much simpler statement (which I think you all should attempt to prove!) is obtained by replacing the word "subbasis" with "basis".

Claim: Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$. Then $(X, \mathcal{T})$ is compact if and only if every open covering of X by open subsets from $\mathcal{B}$ has a finite subcovering.

Exercises from Week 7

Exercise 25. Begin by recalling what the open subsets of X^+ look like (there are two types of open subsets). Check that the preimages of the open subsets under f are open.

You might find Proposition 4.3 and Theorem 9.10 useful along the way.

Exercise 26. I recommend that you begin by proving the following extension of the Hausdroff property, with the help of Lemma 9.11 in the script.

Lemma: A topological space $(X, \mathcal{T})$ is Hausdroff if and only if for any compact subsets $A, B \subseteq X$ with $A \cap B = \varnothing$, there exists open subsets $U, V \in \mathcal{T}$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.

The direction "$\Leftarrow$" is easy. For the direction "$\Rightarrow$", the idea of the proof is the same as what's being done in the proof of Theorem 9.10. To get an intuitive understanding of what's happening, I recommend that you draw a picture in $\mathbb{R}^2$ while you go through the steps of your proof!

Once you've proved the lemma, it's not much work left to solve the exercise!

Bonus exercise: Use the statement of the exercise to prove that any locally compact Hausdroff space X is regular. *Hint:* Think of X as a subspace of the one-point compactification X^+.

Exercise 27. For the first part of the exercise, it's a good idea to use the equivalent definition of T_1 given in the beginning of §12 of the script.

For the second part of the exercise, it turns out that we can *not* weaken the the assumption of T_1 to T_0 in the definition of a normal space. As a counterexample, you might look at the space $X = \{a, b\}$ with $\mathcal{T} = \{\varnothing, X, \{a\}\}$. Check that it's T_0, that any two disjoint closed subsets can be separated by open neighborhoods, and yet, it's not T_1, and therefore not normal.

Exercise 28. Once you've solved the exercise, it might be a good additional exercise to prove a space X is $T_{3.5}$ if and only if it's T_1 and for any $x \in X$ and any closed subset $A \subseteq X$ with $x \notin A$, we can find a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

I personally find this formulation nicer, since it more closely resembles our definition of T_3 and the formulation of Urysohn's lemma.