# PROBLEM SET FOR WEEK 6

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*These problems will be discussed in the exercise classes in Week 6. Hand in clear, independently written solutions to **one** of the problems on Absalon by Thursday, March 16 at 18:00.*

**Problem 1** (Integral extensions)**.**

**(a)** For each of the following algebras, discuss whether it is finite, of finite type and/or integral. Draw a Venn diagram to illustrate your conclusions!

$$\begin{aligned}
&\mathbb{Z} \hookrightarrow \mathbb{Q}\\
&\mathbb{C} \hookrightarrow \mathbb{C}[x]\\
&\mathbb{C}[x] \longrightarrow \mathbb{C}[x,y]/(x^2-y^2)\,, \quad x \mapsto [x]\\
&\mathbb{C}[x] \longrightarrow \mathbb{C}[x,y]/(xy-1)\,, \quad x \mapsto [x]\\
&\mathbb{C} \hookrightarrow \mathbb{C}[x_1,x_2,x_3,\ldots]/(x_1^2,x_2^2,x_3^2,\ldots)
\end{aligned}$$

**(b)** Let $f\colon R \to S$ be a ring homomorphism, and let $n \in \mathbb{N}$ and $r_1,\ldots,r_n \in R$ be such that $(r_1,\ldots,r_n) = R$. Prove that if the induced ring homomorphism $R_{r_i} \to S_{f(r_i)}$ is integral for each $i \in \{1,\ldots,n\}$, then $f$ is integral.

*Hint:* For $s \in S$, consider the ideal of leading coefficients of the polynomials $P \in R[t]$ satisfying $f(P)_*(s) = 0$, and prove that this ideal is all of $R$.

**Problem 2** (Normalization)**.**

**(a)** Prove that every UFD is normal. In other words: if $R$ is a UFD, and $\frac{a}{b} \in \mathrm{Frac}(R)$ is integral with respect to the inclusion $R \hookrightarrow \mathrm{Frac}(R)$, then $\frac{a}{b} \in R$.

**(b)** Determine the normalization of the following rings:

$$\mathbb{C}[x,y]/(x^5-y^3)\,, \quad \mathbb{C}[x,y]/(y-x^2)\,, \quad \mathbb{C}[x,y]/(y^2-x^2-x^3)\,.$$

**(c)** Optional: A domain $R$ is called a *valuation ring* if for any $a \in \mathrm{Frac}(R)$, it holds that either $a \in R$ or $a^{-1} \in R$, if we view $R$ as a subring of $\mathrm{Frac}(R)$. Prove that any valuation ring is a local and normal domain. Also prove the converse is not true. *Hint:* Consider $\mathbb{C}[x,y]_{(x,y)}$.

**Problem 3** ("Lying Over" and "Going Up")**.**

**(a)** For each of the following $\mathbb{C}[x]$-algebras, discuss whether it satisfies the properties in the "Lying Over" theorem, the incomparability theorem (Lemma 14.6) and the "Going Up" theorem:

$$\mathbb{C}[x] \to \frac{\mathbb{C}[x,y]}{(xy-1)}\,, \quad \mathbb{C}[x] \to \frac{\mathbb{C}[x,y]}{(xy)}\,, \quad \mathbb{C}[x] \to \frac{\mathbb{C}[x,y]}{(x(xy-1),y(xy-1))}\,.$$

In all cases, the structure map is given by $x \mapsto [x]$.

**(b)** Let $f\colon R \to S$ be a finite ring homomorphism, such that $S$ is generated by $n$ elements as an $R$-module. Prove that the fibers of the induced map $f^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ have cardinality at most $n$. In other words: prove that for any $\mathfrak{p} \in \mathrm{Spec}(R)$, there are at most $n$ prime ideals $\mathfrak{q} \in \mathrm{Spec}(S)$ lying over $\mathfrak{p}$.

*Hint:* Begin by considering the case when $R$ is a field. Then, consider the case when $\mathfrak{p}$ is a maximal ideal (mod out by $\mathfrak{p}$ to reduce to the first case). Finally, reduce the fully general case to the second case by localizing at $\mathfrak{p}$.