PROBLEM SET FOR WEEK 3

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These problems will be discussed in the exercise classes in Week 3. Hand in clear, independently written solutions to ***one*** *of the problems on Absalon by Thursday, February 23 at 10:00.*

Problem 1 (Localization of rings)**.**

(a) Describe the elements of the ring $\mathbb{C}[x,y]_{(x,y)}$. Give a geometric interpretation!

(b) Prove that $\frac{[y]}{[1]} = 0$ in the ring $\left(\frac{\mathbb{C}[x,y]}{(xy)}\right)_{([x-1],[y])}$.

(c) Let R be a ring, $T \subseteq R$ be a multiplicative subset. Prove that the localization map $\tau\colon R \to R_T$ satisfies the following universal property:

- $\tau(T) \subseteq (R_T)^\times$
- For any ring homomorphism $\varphi\colon R \to S$ such that $\varphi(T) \subseteq S^\times$, there exists a unique ring homomorphism $\widetilde{\varphi}\colon R_T \to S$ that makes the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S\,. \\ {\scriptstyle\tau}\downarrow & \nearrow_{\widetilde{\varphi}} & \\ R_T & & \end{array}$$

Prove also that any R-algebra $\alpha\colon R \to A$ that satisfies this is isomorphic to $\tau\colon R \to R_T$.

(d) Let R be a ring, let $I \subseteq R$ be an ideal, and let $T \subseteq R$ be a multiplicative subset. Let $\pi\colon R \to R/I$ be the quotient map. Prove that $R_T/I_T \cong (R/I)_{\pi(T)}$ as rings.
Hint: Check that $\alpha\colon R/I \to R_T/I_T$, $[r] \mapsto [r/1]$ satisfies the universal property.

Problem 2 (Local rings)**.**

(a) Let R be a ring, and let $\mathfrak{m} \subsetneq R$ be a maximal ideal. Prove that the following are equivalent:

(i) R is a local ring

(ii) $\mathfrak{m} = R \setminus R^\times$

(iii) $1 - f \in R^\times$ for every $f \in \mathfrak{m}$.

Illustrate this with some examples for $R = \mathbb{C}[x]/(x^2)$ and $R = \mathbb{Z}_{(2)}$.

(b) Prove the following simplified version of ***Nakayama's lemma*** (which we will see in greater generality later in the course):

Let R be a local ring with unique maximal ideal $\mathfrak{m}$, and let M be a finitely generated R-module. Then $\mathfrak{m}M = M$ implies $M = 0$.

Hint: Prove that if M is generated by $x_1, \dots, x_n \in M$ for some $n \in \mathbb{N}$, then x_n can be expressed as an R-linear combination of $x_1, \dots, x_{n-1}$.

(c) Let R be a local ring with unique maximal ideal $\mathfrak{m}$, let M be a finitely generated R-module, and let $x_1, \dots, x_n \in M$ for some $n \in \mathbb{N}$. Prove that if $\{[x_1], \dots, [x_n]\}$ generates $M/\mathfrak{m}M$ as an $R/\mathfrak{m}$-vector space, then $\{x_1, \dots, x_n\}$ generates M as an R-module.

Hint: Apply part (b) to the quotient module $M/\operatorname{span}_R\{x_1, \dots, x_n\}$.

(d) Optional: Are the results in (b) and (c) true if we remove the assumption of the module being finitely generated? *Hint:* Try $R = Z_{(2)}$ and $M = \mathbb{Q}/Z_{(2)}$.

Problem 3 (Localization of modules)**.**

(a) Let R be a ring, and let $T \subseteq R$ be a multiplicative subset. Explain what the localization functor $-_T\colon R\mathbf{Mod} \to R_T\mathbf{Mod}$ does to R-modules and R-linear maps, and prove that it really is a functor.

(b) Let R be a ring, let $T \subseteq R$ be a multiplicative set, and let M be an R-module. Prove that the localization map $\gamma\colon M \to M_T$ satisfies the following universal property:

- M_T is T-local, when seen as an R-module.
- For any R-linear map $f\colon M \to N$, where N is a T-local R-module, there exists a unique R-linear map $\widetilde{f}\colon M_T \to N$ that makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N\,. \\ {\scriptstyle\gamma}\big\downarrow & \overset{\widetilde{f}}{\nearrow} & \\ M_T & & \end{array}$$

(c) Use the fact that $-_T\colon R\mathbf{Mod} \to R_T\mathbf{Mod}$ is exact to prove that it sends injective/surjective maps to injective/surjective maps. Also prove that $(M/N)_T \cong M_T/N_T$ for any R-module M and submodule N.

(d) Let M be an R-module, and let $x, y \in M$. Prove that $x = y$ if and only if $\frac{x}{1} = \frac{y}{1}$ in $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subsetneq M$.

Problem 4 (More on the spectrum)**.**

(a) Describe the points and open subsets of $\operatorname{Spec}(\mathbb{C}[[x]])$, where $\mathbb{C}[[x]]$ is the ring of formal power series with complex coefficients.

(b) Let R be a ring and let $T \subseteq R$ be a multiplicative subset. Recall that the localization map $\tau\colon R \to R_T$ induces a bijection

$$\tau^*\colon \operatorname{Spec}(R_T) \to \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \cap T = \varnothing\}\,.$$

Prove that this is a homeomorphism, if the image is given the subspace topology in $\operatorname{Spec}(R)$.

Hint: Prove that τ^* is a closed map onto its image.

(c) What is the image of τ^* when $T = \{f^n : n \in \mathbb{N}\}$ for some $f \in R$, and when $T = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$? Illustrate this with some concrete examples for $R = \mathbb{Z}$.

(d) Optional challenge: Think a bit about what $\operatorname{Spec}(\mathbb{Z}[x])$ looks like. Then have a look at Mumford's famous illustration from *The Red Book of Varieties and Schemes*, and see if you can make sense of some part of it: