PROBLEM SET FOR WEEK 2

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These problems will be discussed in the exercise classes in Week 2.

Hand in solutions to ***one*** *of the problems (i.e. Problem 1, 2, 3 or 4) on Absalon by Thursday, February 16 at 10:00. Your solutions can either be typed or handwritten/scanned, and should be easy to read, with clear, elaborate arguments. You are welcome to discuss the problems with others, but in the end, the solutions should be written down independently, in your own words.*

Problem 1. Let R be a ring, $I \subseteq R$ an ideal, and M an R-module. Let

$$IM = \left\{ \sum_{i=1}^{n} a_i.x_i : n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ x_1, \ldots, x_n \in M \right\}.$$

(a) Prove that $IM \subseteq M$ is an R-submodule, and that M/IM can be given the structure of an R/I-module by letting the R/I-action be given by $[r].[x] = [r.x]$ for $r \in R$ and $x \in M$.

(b) Suppose that M is free over R and that $\{x_1, \ldots, x_n\}$ is a basis. Prove that $\{[x_1], \ldots, [x_n]\}$ is a basis for M/IM over R/I.

(c) Prove that commutative rings have the invariant basis number (IBN) property:

Let M be a free finitely generated R-module, and suppose that $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are bases for M with cardinalities $m, n \in \mathbb{N}$. Then $m = n$.

Hint: Use part (b) together with Krull's theorem. (Treat the special case $R = 0$ separately.)

(d) Optional: Look up an example of a noncommutative ring without the IBN property.

Problem 2. Recall that the ***spectrum*** of a ring R is defined as the set

$$\mathrm{Spec}(R) = \{\text{Prime ideals } \mathfrak{p} \subseteq R\},$$

equipped with the ***Zariski topology***, which is obtained by declaring a subset to be closed if and only if it is of the form

$$\mathbb{V}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I \subseteq \mathfrak{p}\}$$

for some ideal $I \subseteq R$.

(a) Give some examples of points and open subsets in $\mathrm{Spec}(\mathbb{Z})$. Is it a Hausdroff space?

(b) Describe the points and open sets of $\mathrm{Spec}(\mathbb{C}[x]/(x^2 - x))$.

(c) Prove that the sets

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) : f \notin \mathfrak{p}\}$$

for $f \in R$ constitute a base for the topology on $\mathrm{Spec}(R)$.

(d) Prove that $\mathrm{Spec}(R)$ is quasicompact, in the sense that every open cover has a finite subcover. *Hint:* It is enough to check this on an open cover by basic open subsets.

(e) Let $\varphi\colon R \to S$ be a ring homomorphism. Show that it induces a continuous map

$$\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R), \quad \mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

(f) Prove that the assignment in (e) turns $\mathrm{Spec}(-)$ to a contravariant functor from the category of rings to the category of topological spaces (look up what *contravariant* means if you don't already know this!).

Problem 3. Let R be a ring. In this problem, we will study so-called *exact sequences* of R-modules. (If you don't already know what this means, begin by looking it up!)

(a) Explain why there is a unique R-linear map $0 \to M$ and a unique R-linear map $M \to 0$ for any R-module M.

(b) What can be said about an R-linear map $\varphi\colon M \to N$ that fits into an exact sequence of the following form:

(i) $0 \longrightarrow M \xrightarrow{\varphi} N\,$?

(ii) $M \xrightarrow{\varphi} N \longrightarrow 0\,$?

(iii) $0 \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0\,$?

(c) What can be said about an R-module M that fits into an exact sequence

$$0 \longrightarrow M \longrightarrow 0\,?$$

(d) Suppose that R is a field k. Prove that if there exists a short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} L \longrightarrow 0\,,$$

then $N \cong M \oplus L$. Find an example showing that this is not necessarily true over $R = \mathbb{Z}$.

(e) Let P be an R-module. Recall (or look up the fact) that the $\operatorname{Hom}_R(P, -)$ functor takes an R-linear map $\varphi\colon M \to N$ to an R-linear map

$$\varphi_*\colon \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, N)\,, \quad f \mapsto \varphi \circ f\,.$$

Prove that $\operatorname{Hom}_R(P, -)$ is *left exact* in the sense that it takes any short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} L \longrightarrow 0$$

to an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(P, M) \xrightarrow{\varphi_*} \operatorname{Hom}_R(P, N) \xrightarrow{\psi_*} \operatorname{Hom}_R(P, L)\,.$$

Conclude, in particular, that $\operatorname{Hom}_R(P, -)$ preserves injectivity.

(f) Find an example of a ring R, an R-module P and a surjective R-linear map $\psi\colon N \to L$ that is not taken to a surjective R-linear map by $\operatorname{Hom}_R(P, -)$. *Hint:* Try $R = \mathbb{Z}$ and $P = \mathbb{Z}/2$.

Problem 4. In this problem, we will prove a "topological analog" of the classical Nullstellensatz! Let X be a quasicompact Hausdroff space, and consider the ring

$$\mathscr{C}(X) = \{\text{Continuous maps } f\colon X \to \mathbb{R}\}$$

with pointwise addition and multiplication, and the constant map 1 as multiplicative identity.

(a) Prove that $\mathfrak{m}_a := \{f \in \mathscr{C}(X) : f(a) = 0\} \subseteq \mathscr{C}(X)$ is a maximal ideal for any $a \in X$.

(b) Prove that if $a \neq b$, then $\mathfrak{m}_a \neq \mathfrak{m}_b$. *Hint:* Use Urysohn's lemma.

(c) Prove that all maximal ideals in $\mathscr{C}(X)$ are of the form $\mathfrak{m}_a$ for some $a \in X$.
Hint: Assume for a contradiction that there is a maximal ideal $\mathfrak{m} \subsetneq \mathscr{C}(X)$ not of this form, and derive the contradiction that $\mathfrak{m}$ contains a unit.

(d) Conclude that we have a bijective correspondence $X \leftrightarrow \{\text{Maximal ideals of } \mathscr{C}(X)\}$.