PROBLEM SET FOR WEDNESDAY WEEK 1

OSKAR HENRIKSSON

These problems (or a subset of them) will be discussed in the exercise class on Wednesday. Remember that this first week of the course, you don't need to hand in any solutions for grading.

Problem 1.

(a) Let R be a ring, and let $I \subseteq R$ be an ideal. What can be said about R/I if I is maximal, prime or radical? Give some examples for $R = \mathbb{C}[x, y]$.

(b) Give examples of maximal ideals containing $(y - x^2)$ in $\mathbb{C}[x, y]$, as well as maximal ideals containing (12) in $\mathbb{Z}$.

(c) Let R be a ring and $I \subseteq R$ an ideal. Describe the prime ideals of R/I.

(d) Let R and S be rings. What are the prime ideals of $R \times S$? *Hint:* Recall that all ideals of $R \times S$ are of the form $I \times J$, for ideals $I \subseteq R$ and $J \subseteq S$.

(e) Give examples of prime ideals and maximal ideals of the following rings:

$$\mathbb{Z}\,, \quad \mathbb{C}\,, \quad \mathbb{C}[x]\,, \quad \mathbb{Z}[x]\,, \quad \mathbb{C}[[x]] \text{ (the ring of formal power series)}\,.$$

Problem 2. Let k be a field. We will use the following notation:

$$\mathbb{V}(I) = \{a \in k^n : f(a) = 0 \text{ for all } f \in I\} \text{ for } I \subseteq k[x_1, \ldots, x_n]$$
$$\mathbb{I}(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\} \text{ for } X \subseteq k^n\,.$$

(a) Which of the following sets are algebraic subsets of $\mathbb{R}^2$:

$$\varnothing\,, \ \mathbb{R}^2\,, \ \{(\cos(t), \sin(t)) : t \in \mathbb{R}\}\,, \ \{(0, 0), (1, 2)\}\,, \ \{(t, \sin(t)) : t \in \mathbb{R}\}\,, \ \{(x, y) : x^2 + y^2 \leq 1\}\,?$$

(b) Let $I \subseteq J \subseteq k[x_1, \ldots, x_n]$. Prove that $\mathbb{V}(J) \subseteq \mathbb{V}(I)$.

(c) Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Prove that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

(d) Let $X \subseteq k^n$. Prove that $\mathbb{I}(X)$ is a radical ideal.

(e) Prove that $\mathbb{V}(I+J) = \mathbb{V}(I) \cap \mathbb{V}(J)$ and $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$ for ideals $I, J \subseteq k[x_1, \ldots, x_n]$.

(f) Let $X \subseteq k^n$ and consider the ring

$$\mathbb{A}(X) = \{\text{Polynomial functions } f \colon X \to k\}$$

with pointwise addition and multiplication. Prove that $\mathbb{A}(X) \cong k[x_1, \ldots, x_n]/\,\mathbb{I}(X)$.

(g) Later in the course we will see **Hilbert's Nullstellensatz**. A version of the theorem tells us that if k is algebraically closed, then $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$ give a bijective correspondence

$$\{\text{Algebraic subsets } X \subseteq k^n\} \longleftrightarrow \{\text{Radical ideals of } k[x_1, \ldots, x_n]\}\,.$$

Give a counterexample showing that this fails when $k = \mathbb{R}$.

(h) Another consequence of the Nullstellensatz is that if k is algebraically closed, then all maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in k^n$, which gives a bijective correspondance

$$k^n \longleftrightarrow \{\text{Maximal ideals of } k[x_1, \ldots, x_n]\}.$$

Give an example of a maximal ideal in $\mathbb{R}[x]$ that is not of the form $(x - a)$ for $a \in \mathbb{R}$.

Problem 3. Let R be a ring, and let $\operatorname{Nil}(R) = \sqrt{(0)}$ be the nilradical. Prove that the following are equivalent:

(i) R has exactly one prime ideal.

(ii) For any $f \in R$, it holds that f is either a unit or nilpotent.

(iii) The nilradical $\operatorname{Nil}(R)$ is a maximal ideal.

Give some examples of rings with this property.

Problem 4. Let R be a ring, M an R-module, and $N \subseteq M$ a submodule. Prove the following:

(a) If M is finitely generated, then M/N is also finitely generated.

(b) If M/N and N are finitely generated, then M is also finitely generated.

(c) Give an example that shows that M being finitely generated does *not* imply that N is finitely generated. *Hint:* Try $R = M = k[x_1, x_2, \ldots]$.