Chapter 3C: From fans to normal toric varieties

Oskar Henriksson
University of Copenhagen

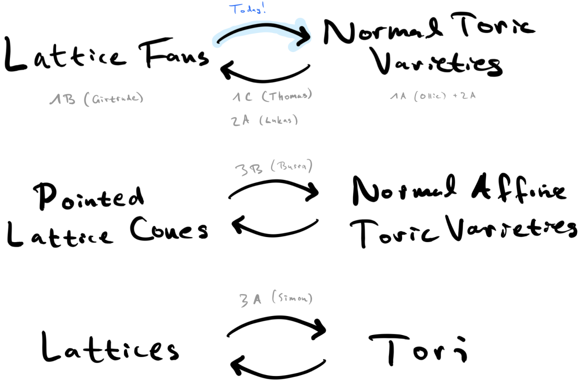
The Ghent University weekly
reading seminar in Toric Geometry

April 19, 2021

Main results this far...

We have constructed functors back and forth between the combinatorial world of *lattices and fans*, and the geometric world of *tori and toric varieties*.

It turns out that these are *equivalences of categories*:



Agenda for today

1. Review from last week
2. Construction
3. Properties of the construction
4. Examples

References

[Hau] J. Hausen, *Video Course on Toric Varieties*, 2020. Lecture notes available at `https://www.math.uni-tuebingen.de/user/hausen/TV-Video-Course/tv-video-course.pdf`.

[CLS] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, American Mathematical Society, 2011.

[Cox] D. A. Cox, *Lectures on Toric Varieties*, 2005. Lecture notes available at `https://dacox.people.amherst.edu/`.

[MS] M. Michałek and B. Sturmfels, *Invitation to Nonlinear Algebra*, American Mathematical Society, 2021.

[Gat] A. Gathmann, *Class notes on Algebraic Geometry*, 2020. Lecture notes available at `https://www.mathematik.uni-kl.de/~gathmann/de/alggeom.php`.

Section 1

Review from previous week

From cones to affine toric varieties

Let $N \cong \mathbb{Z}^n$ be a lattice, with dual lattice M.

Let σ be a pointed cone in N.

Set $\mathbb{T}_N = \mathrm{Spec}(\mathbb{C}[M])$ and $X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$.

Let $x_0 \in X_\sigma$ correspond to the maximal ideal $(\, \chi^u - 1 : u \in \sigma^\vee \cap M\,)$.

Theorem

$(X_\sigma, \mathbb{T}_N, x_0)$ is an affine toric variety.

Sketch of proof.

- $\mathbb{C}[\sigma^\vee \cap M]$ is a finitely generated $\mathbb{C}$-algebra by Gordan's lemma.
- $\mathbb{C}[\sigma^\vee \cap M] \subseteq \mathbb{C}[M] \cong \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$ is an integral domain.
- We get an action $\mathbb{T}_N \times X_\sigma \to X_\sigma$ induced by $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[M] \otimes \mathbb{C}[\sigma^\vee \cap M]$, $\chi^u \mapsto \chi^u \otimes \chi^u$.

Example

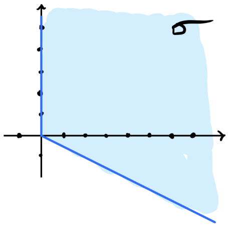
Let $N = \mathbb{Z}^2$, $M \cong \mathbb{Z}^2$ via usual inner product.

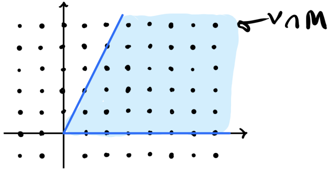
$$\sigma = \text{cone}\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}, \quad \sigma^\vee = \text{cone}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\},$$

$$\sigma^\vee \cap M = \mathbb{N}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$$

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x, xy, xy^2] \cong \frac{\mathbb{C}[z_1, z_2, z_3]}{(z_2^2 - z_1 z_3)}, \quad X_\sigma \cong V(z_2^2 - z_1 z_3) \subseteq \mathbb{C}^3.$$

$$(\mathbb{C}^*)^2 \circlearrowright X_\sigma \text{ via } (t_1, t_2).(z_1, z_2, z_3) = (t_1 z_1, t_1 t_2^2 z_2, t_1 t_2 z_3).$$





Normality of affine toric varieties

Proposition

X_σ is a *normal* affine variety for any pointed lattice cone (σ, N).

Sketch of proof (see [Cox, Thm. 1.13] for details).

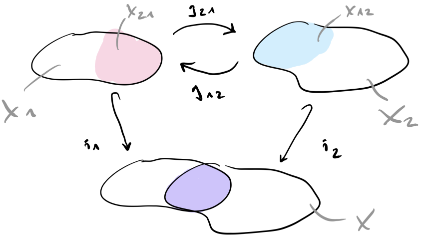
- It suffices to show $\mathbb{C}[X_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ is *integrally closed*.
- Suppose $\sigma = \mathrm{cone}(v_1, \ldots, v_r)$ for minimal generators $v_1, \ldots, v_r \in N$. Set $\tau_i = \mathrm{cone}(v_i)$. Then $\mathbb{C}[\sigma^\vee \cap M] = \bigcap_{i=1}^r \mathbb{C}[\tau_i^\vee \cap M]$.
- Note that $\mathbb{C}[\tau_i^\vee \cap M] \cong \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm]$, which is integrally closed (it's even a UFD).
- The result now follows from the fact that the rings $\mathbb{C}[\tau_i^\vee \cap M]$ have the same field of fraction.

Section 2

The construction

Gluing of two affine varieties

Let X_1 and X_2 be affine varieties, with isomorphic open subsets $X_{21} \subseteq X_1$ and $X_{12} \subseteq X_2$ as illustrated in the picture:



The gluing of X_1 and X_2 via g_{21} and g_{12} is defined as the quotient

$$X = \frac{X_1 \sqcup X_2}{a \sim g_{21}(a) \ \forall\, a \in X_1},$$

with sheaf $\mathscr{O}_X(U) = \{\varphi \colon U \to \mathbb{C} : i_k^* \varphi \in \mathscr{O}_{X_k}(i_k^{-1}(U)) \text{ for } k = 1, 2\}$.

Gluing (general case)

- Let $\{x_\alpha\}_{\alpha\in J}$ be a finite collection of affine varieties.
- Suppose that for each $\alpha, \beta \in J$, we have open subsets $X_{\alpha\beta} \subseteq X_\beta$, $X_{\beta\alpha} \subseteq X_\alpha$, and mutually inverse isomorphisms

$$\begin{tikzcd} X_{\alpha\beta} \arrow[r, bend left, "g_{\alpha\beta}"] & X_{\beta\alpha} \arrow[l, bend left, "g_{\beta\alpha}"] \end{tikzcd}.$$

- For each $\alpha, \beta, \gamma \in J$, it holds that $g_{\beta\alpha}(X_{\beta\alpha} \cap X_{\gamma\alpha}) = X_{\alpha\beta} \cap X_{\gamma\beta}$, and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on this set.
- We then define the gluing as

$$X = \frac{\coprod_{\alpha\in J} X_\alpha}{a \sim g_{\beta\alpha}(a) \ \forall \alpha, \beta \in J, a \in X_{\beta\alpha}}$$

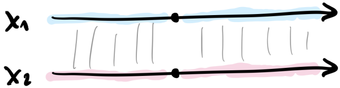
Warning: Gluings might not be separated

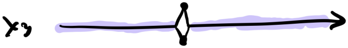
It is easy to verify that the gluing of affine varieties gives a prevariety, but it will not necessarily be *separated* in the following sense:

Definition

A prevariety X is said to be a *variety* (or a *separated variety*) if $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \times X$ (in the category of prevarieties).

Classical example: $X_1 = X_2 = \mathbb{C}$, $X_{21} = X_{12} = \mathbb{C}^*$, with $g_{21} = g_{12} = \mathrm{id}_{\mathbb{C}^*}$. Then the gluing X is the affine line with two origins.





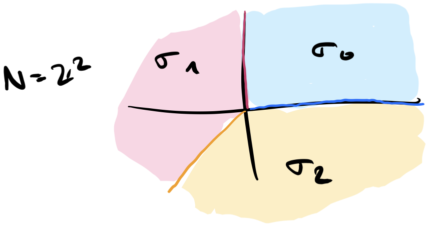
Quick review of fans

Definition

A *fan* Σ in a lattice N is a collection of pointed cones in $N_{\mathbb{Q}}$, such that

1. $[\sigma \in \Sigma \text{ and } \tau \preccurlyeq \sigma] \Longrightarrow \tau \in \Sigma$.
2. $[\sigma, \sigma' \in \Sigma] \Longrightarrow [\sigma \cap \sigma' \preccurlyeq \sigma \text{ and } \sigma \cap \sigma' \preccurlyeq \sigma']$.

Example:



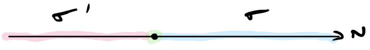
The construction

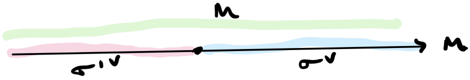
Let Σ be a fan in a lattice $N \cong \mathbb{Z}^n$, with dual lattice $M \cong \mathbb{Z}^n$.

1. Each $\sigma \in \Sigma$ gives rise to an affine toric variety $X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, with an action of the torus $\mathbb{T}_N = \mathrm{Spec}(\mathbb{C}[M])$.
2. Suppose $\tau \preccurlyeq \sigma$. Then $\tau = u^\perp \cap \sigma \implies \tau^\vee = \sigma^\vee + \mathbb{Q}u$
   $\implies \tau^\vee \cap M = \sigma^\vee \cap M + \mathbb{Z}u \implies \mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[\sigma^\vee \cap M]_{\chi^u}$.
   This means $X_\tau = (X_\sigma)_{\chi^u}$.
3. If $\tau = \sigma_1 \cap \sigma_2$, then $u^\perp \cap \sigma_1 = \tau = (-u)^\perp \cap \sigma_2$
   for some $u \in \sigma_1^\vee \cap (-\sigma_2^\vee)$. This gives
   $$X_{\sigma_1} \supseteq (X_{\sigma_1})_{\chi^u} = X_\tau = (X_{\sigma_2})_{\chi^{-u}} \subseteq X_{\sigma_2}\,.$$
4. Glue $\{X_\sigma\}_{\sigma \in \Sigma}$ to a (pre)variety X_Σ via corresponding gluing maps
   $$\begin{tikzcd} (X_{\sigma_1})_{\chi^u} \arrow[r, bend left, "g_{\sigma_1\sigma_2}"] & (X_{\sigma_2})_{\chi^{-u}} \arrow[l, bend left, "g_{\sigma_2\sigma_1}"] \end{tikzcd}.$$

Example: The projective line

Let $N = \mathbb{Z}$ and $\Sigma = \{\sigma, \sigma, \{0\}\}$ as in the picture below:





Then $\mathbb{C}[(\sigma')^\vee \cap M] = \mathbb{C}[x^{-1}]$, $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x]$ and $\mathbb{C}[(\sigma' \cap \sigma)^\vee \cap M] = \mathbb{C}[x^\pm]$.

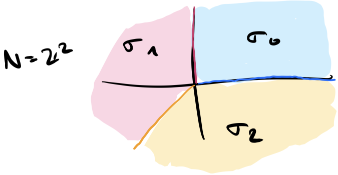
We're gluing two copies of $\mathbb{C}$ along $\mathbb{C}^*$.

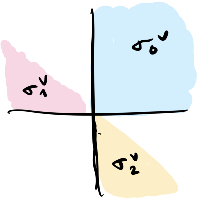
Are the gluing maps the same as in the usual construction of $\mathbb{P}^1$? Yes!

$$\begin{tikzcd}
x^{-1} \arrow[d, mapsto] & \mathbb{C}[x^{-1}] \arrow[r, hook] \arrow[d, "\cong"'] & \mathbb{C}[x^\pm] \arrow[d, equal] & \mathbb{C}[x] \arrow[l, hook'] \arrow[d, equal] \\
x & \mathbb{C}[x] \arrow[r] & \mathbb{C}[x^\pm] & \mathbb{C}[x] \arrow[l, hook'] \\
& x \arrow[r, mapsto] & x^{-1} &
\end{tikzcd}$$

Example: The projective plane

Let $N = \mathbb{Z}^2$ and let Σ be as below.





$$X_{\sigma_0} = \operatorname{Spec}(\mathbb{C}[x, y]) \cong \mathbb{C}^2$$
$$X_{\sigma_1} = \operatorname{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \cong \mathbb{C}^2$$
$$X_{\sigma_2} = \operatorname{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]) \cong \mathbb{C}^2$$
$$X_{\sigma_0 \cap \sigma_2} = \operatorname{Spec}(\mathbb{C}[x, y^{\pm 1}]) \cong \mathbb{C} \times \mathbb{C}^*$$

Properties of the construction

Proposition

X_Σ is a toric variety for all lattice fans (Σ, N)

Sketch of proof (see [CLS] for more details).

- We know that we have an open embedding $\mathbb{T}_N \hookrightarrow X_\sigma$ [since $\{0\} \preccurlyeq \sigma$] and an action $\mathbb{T}_N \circlearrowright X_\sigma$ for each $\sigma \in \Sigma$.
- The embedded tori get identified when we glue, and the action is respected by the inclusions $X_{\sigma_1} \hookleftarrow X_{\sigma_1 \cap \sigma_2} \hookrightarrow X_{\sigma_2}$, so we get a global action $\mathbb{T}_N \circlearrowright X_\Sigma$, and an open dense embedding $\mathbb{T}_N \hookrightarrow X_\Sigma$.
- In particular, X_Σ is *irreducible* and n-dimensional.
- Since $\{0\} \preccurlyeq \sigma$, we have an open embedding $\mathbb{T}_N \subseteq X_\sigma$. These embedded tori get identified when we glue, so we have an open embedding $\mathbb{T}_N \subseteq X_\Sigma$.

Properties of the construction (cont.)

Proposition

X_Σ is a toric variety for all lattice fans (Σ, N)

Sketch of proof (cont.).

- For *separatedness*, we want to show that the diagonal map $\Delta \colon X_\Sigma \to X_\Sigma \times X_\Sigma$ has closed image.
- Suffices to show that $\Delta \colon X_\tau \to X_{\sigma_1} \times X_{\sigma_2}$, $x \mapsto (i_1(x), i_2(x))$ has closed image.
- On the level of coordinate rings:
  $\Delta^* \colon \mathbb{C}[\sigma_1^\vee \cap M] \otimes \mathbb{C}[\sigma_2^\vee \cap M] \to \mathbb{C}[\tau^\vee \cap M], \quad \chi^u \otimes \chi^v \mapsto \chi^{u+v},$
  which is surjective by the *separation lemma* [CLS, Prop. 3.1.3].
- It then follows that $\Delta(X_\tau) = V(\ker(\Delta^*))$, i.e. it is closed in $X_{\sigma_1} \times X_{\sigma_2}$.
- *Normality* follows from normality of the X_σ's.

Functoriality

Definition

A *map of lattice fans* $F\colon (\Sigma, N) \to (\Sigma', N')$ is a group homom. $F\colon N \to N'$ such that for every $\sigma \in \Sigma$, it holds that $F(\sigma) \subseteq \sigma'$ for some $\sigma' \in \Sigma'$.

Recall from last week, that F induces a toric morphism $\varphi_F\colon X_\sigma \to X_{\sigma'}$ whenever $F(\sigma) \subseteq \sigma'$, so we have a mapping $\coprod_{\sigma\in\Sigma} X_\sigma \to \coprod_{\sigma'\in\Sigma'} X_{\sigma'}$.

Suppose $F(\sigma) \subseteq \sigma_1' \cap \sigma_2'$. Then $X_{\sigma_1'}$ and $X_{\sigma_2'}$ are glued along $X_{\sigma_1'\cap\sigma_2'}$ in such a way that $\varphi_{\sigma_1,\sigma_1'}(p)$ and $\varphi_{\sigma_1,\sigma_2'}(p)$ are identified.

It turns out that we well-defined map toric morphism $\varphi_F\colon (X_\Sigma, \mathbb{T}_N, x_0) \to (X_{\Sigma'}, \mathbb{T}_{N'}, x_0')$, and that the construction is functorial.

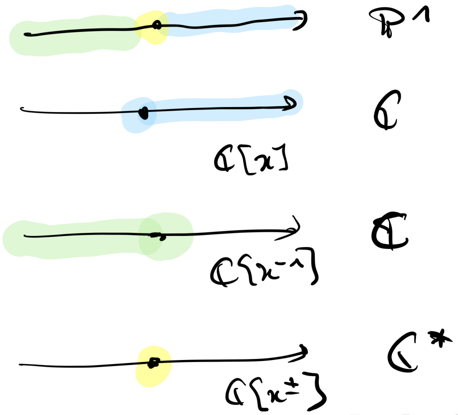
The main theorem

$\Sigma \mapsto X_\Sigma$ is an equivalence of categories between lattice fans and normal toric varieties, with inverse functor $\Sigma(-)$.

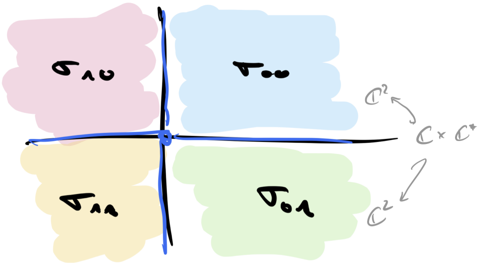
A simple application

Proposition

Every 1-dimensional normal toric variety is isomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$.



Example: $\mathbb{P}^1 \times \mathbb{P}^1$



Note:

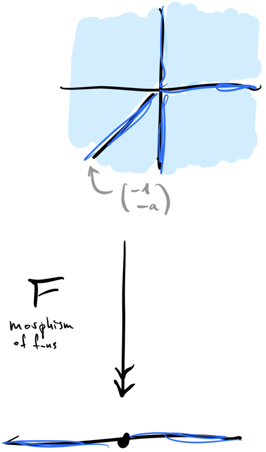
$X_{\sigma_{ij}} \cong \mathcal{U}_i \times \mathcal{U}_j$ in $\mathbb{P}^1 \times \mathbb{P}^1$

Proposition 3.1.14 in [CLS]

For any two fans Σ and Σ', it holds that $X_\Sigma \times X_{\Sigma'} \cong X_{\Sigma \times \Sigma'}$.

Both the product of fans and the product of toric varieties are *categorical* products, so this follows from the equivalence of categories!

Example: Hirzebruch surface

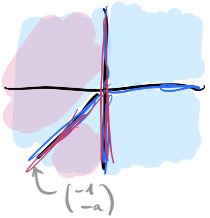


$$\begin{array}{c}(\Sigma_a, \mathbb{T}^2)\\ \Big\downarrow \varphi_F \\ (\mathbb{P}^1, \mathbb{T}^1)\end{array}$$

morphism of toric varieties

Example: Hirzebruch surface

$\mathbb{C} \times \mathbb{P}^1$



$$(\Sigma_a, \mathbb{T}^2)$$

F
morphism of fans

φ_F
morphism of toric varieties

Ruled surface over $\mathbb{P}^1$!

$\mathbb{C}$

$$(\mathbb{P}^1, \mathbb{T}^1)$$

Coming up...

We will see many more examples of how properties of a lattice fan (Σ, N) relates to the properties of the associated toric variety $(X_\Sigma, \mathbb{T}_N)$.

For instance:

- The cones in Σ correspond to $\mathbb{T}_N$ orbits of X_Σ (Ch. 4).
- The variety X_Σ is complete iff the cones of Σ cover all of $N_\mathbb{Q}$ (Ch. 5).
- A cone $\sigma \in \Sigma$ gives rise to a smooth variety X_σ iff the primitive generators of σ extend to a $\mathbb{Z}$-basis for N (Ch. 6).