Calculus: True or False?

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Prove or disprove the following statements. Also ask yourself if any of the false statements are "almost true," in the sense that they can be reformulated into similar, but true, statements.

(1) The square of any complex number is a real number.

(2) For every $z \in \mathbb{C}$ there exists a *multiplicative inverse*, i.e. a $w \in \mathbb{C}$ such that $zw = 1$.

(3) The sum of two rational numbers is always rational, and the sum of two irrational numbers is always irrational.

(4) For every $z \in \mathbb{C}$ there exists an $n \in \mathbb{Z}^+$ such that $z^n \in \mathbb{R}$.

(5) Limits are unique in the following sense: Let $f\colon \mathbb{R} \supseteq D \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. If two real numbers L and L' both satisfy the definition of being a limit of $f(x)$, as $x \to a$, then $L' = L$.

(6) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} f(x)g(x) = \infty$.

(7) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \big(f(x) - g(x)\big) = 0$.

(8) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous and injective function. Then f is strictly increasing or strictly decreasing.

(9) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions such that $f' = g'$. Then $f = g$.

(10) Suppose that $F\colon \mathbb{R} \to \mathbb{R}$ is a primitive function of both $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$. Then $f = g$.

(11) Every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable.

(12) Every differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(13) Every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ has a primitive function.

(14) If a function $f\colon \mathbb{R} \to \mathbb{R}$ has a primitive function, then f is continuous.

(15) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then $\frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_a^b f(x)\,\mathrm{d}x\Big) = f(x)$.

(16) Let $D \subseteq \mathbb{R}$ and let $f\colon D \to \mathbb{R}$ be a differentiable function with $f'(x) = 0$ for all $x \in D$. Then f is constant on all of D.

(17) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and suppose that we want to determine the limit of $f(x)$ as $x \to 0^+$.

Suppose that we know that $f(1) = 0$ $f(0.1) = 0$, $f(0.01) = 0$, $f(0.001) = 0$ and so on [i.e. that $f(10^{-n}) = 0$ for all $n \in \mathbb{N}$]. From this we can conclude that $f(x) \to 0$ as $x \to 0^+$.

(18) We can approximate e arbitrarily well by elements of the sequence $(s_1, s_2, s_3, \ldots)$, where $s_n = \sum_{k=1}^{n} \frac{1}{k!}$, and we can approximate $\sin(e)$ arbitrarily well by the sequence $(\sin(s_1), \sin(s_2), \sin(s_3), \ldots)$.

(19) If $\int_0^\infty f(x)\,\mathrm{d}x$ and $\int_0^\infty g(x)\,\mathrm{d}x$ both diverge, then $\int_0^\infty \big(f(x) + g(x)\big)\,\mathrm{d}x$ also diverges.

(20) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be integratable functions such that $f(x) \leqslant g(x)$ for all $x \in \mathbb{R}$. If $\int_0^\infty f(x)\,\mathrm{d}x$ diverges, then $\int_0^\infty g(x)\,\mathrm{d}x$ also diverges.

(21) Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integratable function, and let $\lambda \in \mathbb{R}$. If $\int_0^\infty f(x)\,\mathrm{d}x$ converges, then $\int_0^\infty \lambda f(x)\,\mathrm{d}x$ also converges.

(22) Let $D \subseteq \mathbb{R}$ and let $f\colon D \to \mathbb{R}$ be a function. If $F\colon D \to \mathbb{R}$ and $G\colon D \to \mathbb{R}$ are both primitive functions of f, then F and G only differs by a constant, i.e. there exists a $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in D$.

(23) A series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $a_k \to 0$ as $k \to \infty$.

(24) If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

(25) Let $f\colon \mathbb{R} \to \mathbb{R}$ be infinitely differentiable at all points, and suppose that $f^{(k)}(0) = 0$ for all $k \in \mathbb{Z}_0^+$. Then $f(x) = 0$ for all $x \in \mathbb{R}$.