“The miracle of *u*-substitutions”

Why and how does it work?

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November 12, 2017

A first example

$$\int \cos(x^2 + 1) \cdot 2x \, dx$$

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$$\int \cos(x^2 + 1) \cdot 2x \, \mathrm{d}x$$

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$$\int \cos(x^2 + 1) \cdot 2x \, dx$$

$$= \sin(x^2 + 1) + C$$

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$$\int \cos(x^2 + 1) \cdot 2x \, \mathrm{d}x$$

$$= \int \cos(u) \, \mathrm{d}u$$

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$$= \int \cos(u) \, du$$

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In general

$$\int f(\,g(x)\,) \cdot g'(x)\,\mathrm{d}x$$

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Another example

$$\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$$

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$$= \frac{1}{1 + (e^x)^2} \cdot e^x\, \mathrm{d}x$$

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$$= \arctan(e^x) + C$$

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$$= \arctan(e^x) + C$$

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$$= \arctan(e^x) + C$$

A final example

$$\int x\sqrt{x+1}\,\mathrm{d}x$$

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$$= \int (u^{3/2}-u^{1/2})\,\mathrm{d}u$$

$$= \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C$$

$$= ???$$

A final example

$$
\begin{aligned}
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=\ &\int (x+1-1)\sqrt{x+1}\cdot 1\,\mathrm{d}x \\
=\ &\int (u-1)\sqrt{u}\,\mathrm{d}u \\
=\ &\int \left(u^{3/2}-u^{1/2}\right)\mathrm{d}u \\
=\ &\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}+C \\
=\ &\tfrac{2}{5}(x+1)^{5/2}-\tfrac{2}{3}(x+1)^{3/2}+C
\end{aligned}
$$