Some Aspects of Elementary Lie Theory

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Outline

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Section 1

Basic definitions and the Big Idea

What is a Lie Group?

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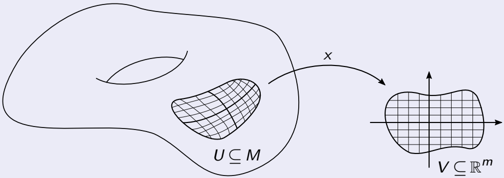

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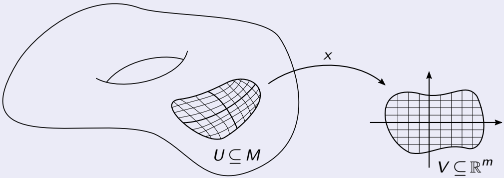

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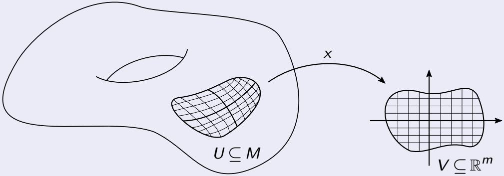

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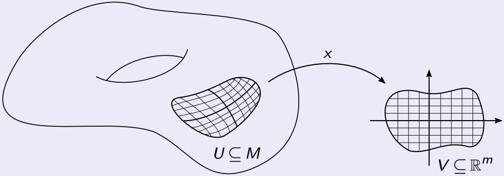

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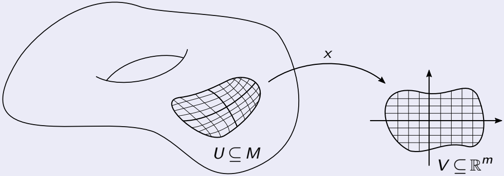

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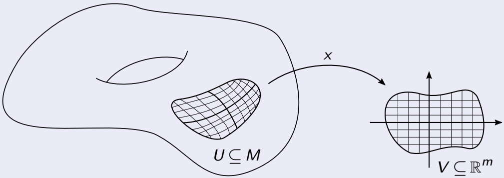

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- Existence of inverses: $\forall a \in G \, \exists a^{-1} \in G : a \cdot a^{-1} = a^{-1} \cdot a = 1$.

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Example

The unit circle $S^1 \subseteq \mathbb{C}$ is a smooth curve with a smooth group operation.

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- $\left\| \begin{pmatrix} 2+i & 5 \\ i & 1-2i \end{pmatrix} \right\| = \sqrt{2^2+1^2+1^2+5^2+1^2+(-2)^2} = \sqrt{36} = 6$

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- The determinant $\det\colon \mathbb{C}^{n\times n} \to \mathbb{C}$ is continuous, because $\det(A)$ is a polynomial in the real and imaginary parts of the coefficients of A.

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- $(\mathbb{R}, +) \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$.

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Counterexample (Birkhoff, 1936)

$$\underbrace{\underbrace{\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}}_{\text{Matrix Lie group}} \Big/ \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}}_{\text{Lie group but } \textit{can't} \text{ be realized as a matrix Lie group}}$$

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- The study of (continuous) linear actions of groups on vector spaces is called *representation theory*.

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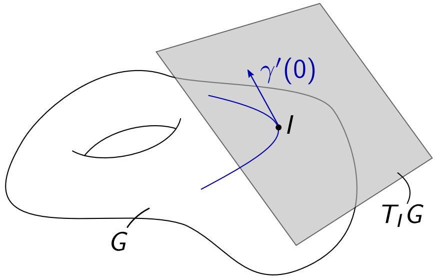

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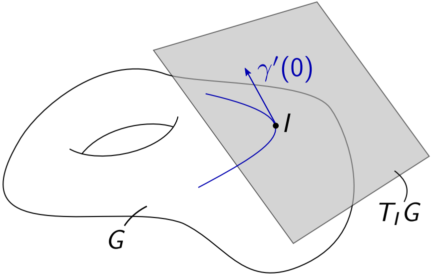

Idea of Lie Theory

We can use linear algebra to study G **via its tangent space at the identity**, provided that G is "sufficiently connected."

Section 2

Toolbox: Exponential map and Lie algebras

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- $\frac{d}{dt} e^{tX} = X\, e^{tX}$.

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Claim: $e^X e^Y = e^{X+Y}$ if $XY = YX$.

Proof.

$$e^X e^Y = \left(\sum_{k=0} \frac{X^k}{k!}\right)\left(\sum_{k=0} \frac{Y^k}{k!}\right) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{X^k}{k!}\frac{Y^{m-k}}{(m-k)!}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k} X^k Y^{m-k} = \sum_{m=0}^{\infty}\frac{(X+Y)^m}{m!} = e^{X+Y}.$$

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Claim: $[e^X]^{-1} = e^{-X}$.

Proof. X and $-X$ commute, so

$$e^X \cdot e^{-X} = e^{-X} \cdot e^X = e^{X+(-X)} = e^0 = I.$$

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$$(\varphi(t))_{ij} = (e^{tX})_{ij} = \sum_{k=0}^{\infty} \frac{t^k (X^k)_{ij}}{k!}$$

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$$\begin{aligned}\log(\exp(X)) &= \log\left(I + X + \tfrac{1}{2!}X^2 + \cdots\right)\\ &= \left(X + \tfrac{1}{2!}X^2 + \cdots\right) - \tfrac{1}{2}\left(X + \tfrac{1}{2!}X^2 + \cdots\right)^2 + \cdots\end{aligned}$$

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Tool #2: The Lie Algebra of a Matrix Lie Group

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A *Lie algebra* is a vector space $\mathbf{g}$ over some field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, equipped with a binary operation $[.,.]\colon \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ that satisfies the following conditions:

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- Bilinearity: $[aX + bY, z] = a[X, Z] + b[Y, Z]$ and $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for all $X, Y, Z \in \mathbf{g}$ and $a, b \in \mathbb{F}$.

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Theorem 2.52

The Lie algebra $\mathbf{g}$ of a matrix Lie group G equals $T_I G$.

Tool #2: The Lie Algebra of a Matrix Lie Group (cont.)

Example: The Lie algebra of $\mathbf{SO}(n)$

$$\mathbf{so}(n) = \{X \in \mathbb{R}^{n\times n} : X^\top + X = 0\}$$

Tool #2: The Lie Algebra of a Matrix Lie Group (cont.)

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Tool #3: Facts about how G and $\mathfrak{g}$ are related

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Theorem 2.45

Let G be a matrix Lie group with Lie algebra $\mathbf{g}$. Then there exists an open neighbourhood U of 0 in $\mathbf{g}$, and an open neighbourhood V of I in G, such that

$$\exp\big|_U \colon U \to V$$

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Theorem 2.46

Let G be a **path-connected** matrix Lie group with Lie algebra $\mathbf{g}$. Then for any $A \in G$, there exists some $X_1, X_2, \ldots, X_m \in \mathbf{g}$ such that

$$A = e^{X_1} e^{X_2} \cdots e^{X_m}.$$

Sidenote: The Role of the Lie Bracket

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Let $A, B \in G$. By previous theorem, $A = e^{X_1} \cdots e^{X_m}$ and $B = e^{Y_1} \cdots e^{Y_k}$, so we get

$$\begin{aligned} A \cdot B &= e^{X_1} \cdots e^{X_m} e^{Y_1} \cdots e^{Y_k} \\ &= e^{X_1 + \cdots + X_m + Y_1 + \cdots + Y_k}. \end{aligned}$$

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In this case the Lie bracket on $\mathbf{g}$ is not needed! It's also very uninteresting:

$$[X, Y] = XY - YX = 0 .$$

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Remark. G inherits the commutativity of $\mathbf{g}$ in this example.

Sidenote: The Role of the Lie Bracket (cont.)

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Example of how the Lie bracket helps us understand multiplication in the group:

Theorem (Baker–Campbell–Hausdorff)

For $X, Y \in \mathbf{g}$ sufficiently close to 0, it holds that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots},$$

where "$\cdots$" denotes a convergent series of nested Lie brackets terms.

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We now have three main tools that we can use for studying a matrix Lie group G:

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1. A locally invertible matrix exponential.

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1. A locally invertible matrix exponential.
2. A Lie algebra: the vector space $T_I G$ equipped with a Lie bracket.
3. Knowledge of a locally 1-to-1 correspondance between G and $\mathbf{g}$, and the fact that $\mathbf{g}$ generates G if G is path-connected.

Section 3

Key result: Lie group vs. Lie algebra homomorphisms

Lie Homomorphisms

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A smooth (continuous) map $\Phi\colon G \to H$ between Lie groups, such that

$$\Phi(AB) = \Phi(A)\Phi(B) \text{ for all } A, B \in G\,,$$

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A linear map $\varphi \colon \mathbf{g} \to \mathbf{h}$ between Lie algebras (over the same field) is said to be a *Lie algebra homomorphism* if

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Lie group homomorphisms $\rightarrow$ Lie algebra homomorphisms

Lie group homomorphisms $\to$ Lie algebra homomorphisms

Theorem 2.56

Let G and H be matrix Lie groups, with Lie algebras $\mathbf{g}$ and $\mathbf{h}$, respectively. Let $\Phi\colon G \to H$ be a Lie group homomorphism. Then there exists a (unique) Lie algebra homomorphism $\varphi\colon \mathbf{g} \to \mathbf{h}$ that makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & H \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathbf{g} & \xrightarrow{\varphi} & \mathbf{h} \end{array}$$

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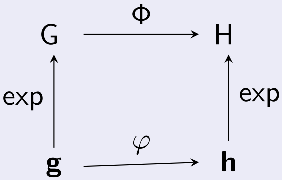

It can be computed using the formula

$$\varphi(X) = d\Phi_I(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0}.$$

Simply connectedness

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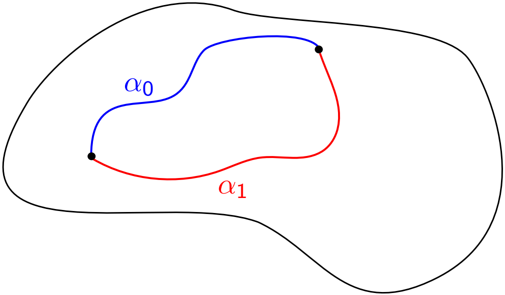

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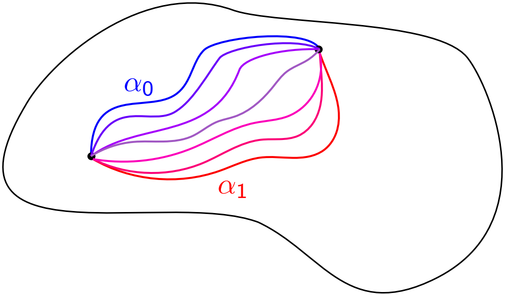

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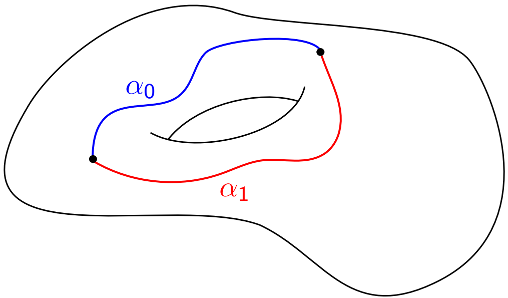

Not simply connected!

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Theorem 2.60

Let G and H be as above. If G is **path-connected** and **simply connected**, then for every Lie algebra homomorphism $\varphi\colon \mathbf{g} \to \mathbf{h}$ there exists a (unique) Lie group homomorphism $\Phi\colon G \to H$ such that $d\Phi_I = \varphi$, i.e. such that the following diagram commutes:

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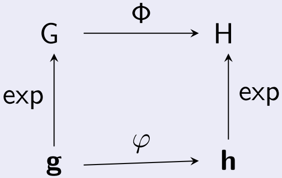

Corollary (Chapter 3)

There is a 1-to-1 correspondence between the ways a **path-connected, simply connected** matrix Lie group G can act continuously and linearly on finite-dimensional vector spaces, and the ways its Lie algebra $\mathbf{g}$ can act continuously and linearly on finite-dimensional vector spaces.

Section 4

Case study: SO(3)

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Let B^3 be the closed ball in $\mathbb{R}^3$ of radius π, and let $\sim$ be the equivalence relation on B^3 defined by $u \sim u$ for all $u \in B^3$ and $u \sim (-u)$ iff $|u| = \pi$. Then $\mathbf{SO}(3) \simeq B^3/\sim$

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Sketch of proof. Consider $f\colon B^3/{\sim} \to \mathbf{SO}(3)$ defined by

$$f(u) = \begin{cases} I & \text{if } u = 0 \\ R_{u/\|u\|,\|u\|} & \text{if } u \neq 0\,. \end{cases}$$

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- Because $\mathbf{SO}(3)$ is not simply connected, we have to be careful when we translate the representations of $\mathbf{so}(3)$ to representations of $\mathbf{SO}(3)$.
- It turns out that it is only the odd-dimensional representations of $\mathbf{so}(3)$ that correspond to representations of $\mathbf{SO}(3)$.

Timeline